On the Asymptotic Efficiency of Distributed Estimation Systems with Constant Modulus Signals over Multiple-Access Channels

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Abstract

A distributed estimation problem is considered with multiple-access channels between sensors and a fusion center. The sensors phase-modulate their noisy observations before transmitting them to the fusion center, where a signal parameter is estimated. The asymptotic efficiency of this estimator is then determined by using two inequalities that relate the Fisher information and the characteristic function. A necessary and sufficient condition for equality is found for the first time in the literature. The loss in efficiency of the distributed estimation scheme relative to the centralized approach is quantified for different sensing noise distributions. It is shown that the distributed estimator does not incur an efficiency loss if and only if the sensing noise distribution is Gaussian.

Index Terms

Fisher information, characteristic function, asymptotic efficiency, wireless sensor network, distributed estimation

I. Introduction

In distributed estimation, multiple sensors observe noisy data that depend on a parameter, and after some processing, transmit this information to a fusion center (FC) over wireless communication channels, where the parameter is estimated. The communication channels between the

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sensors and FC may be orthogonal [1], in which case the FC receives data from each sensor individually, or multiple-access [2], where the FC receives the sum of all sensor transmissions. Transmissions from the sensors to the FC may be digital or analog. In the case of analog transmissions, the most commonly used technique is the amplify-and-forward scheme, where the instantaneous transmit power depends on the individual sensor observations and can be arbitrarily high.

In this paper, we consider a distributed estimation problem with sensors transmitting observations to a fusion center over Gaussian multiple-access channels [3]. The sensors observe a parameter in noise, and phase modulate their observations before transmitting them to the FC. These analog transmissions are appropriately pulse-shaped to consume finite bandwidth. The sum of the phase modulated data is received at the FC due to the multiple-access channels between the sensors and the FC. As the number of sensors increases, the signal received at the FC approximates the characteristic function of the sensed data. The structure of the characteristic function is exploited to estimate the location parameter of the data being observed.

The efficiency of this scheme was not considered in [3]. Towards this goal, we investigate the relationship between the Fisher information about a location parameter and the characteristic function of the additive noise by providing a new derivation for two inequalities that involve the Fisher information and the characteristic function. These inequalities were originally derived using a different approach and applied in a quantum physics setting to estimate the survival probability of a quantum state in [4]. Conditions for equality are also delineated herein for the first time in the literature.

These results are used to investigate the asymptotic efficiency of the distributed estimation scheme, employing phase modulation for transmitting data over a Gaussian multiple-access channel. Since the phase modulation is a part of the system design, we include the effect of phase modulation on the relative efficiency of the estimator. It is shown that there is no loss in efficiency if and only if the sensing noise is Gaussian.

The rest of this paper is organized as follows. The relationship between the Fisher information and the characteristic function is examined in Section II. The distributed estimation problem is introduced in Section III. The asymptotic relative efficiency for each of several different sensing noise distributions is calculated in Section III-C. In Section IV, simulations are used to verify our results. Finally, concluding remarks are presented in Section V.

II. The Inequalities

Consider a model where a deterministic location parameter, θ, is related to observations $x_l = \theta + \eta_l$, $l = 1, \ldots, L$, where η_l are iid and real-valued random variables. Let the characteristic function of η_l be $\varphi(\omega) := E[e^{j\omega\eta_l}]$ and let the Fisher information be defined as [5], [6]

$$I(\eta) := \int_{-\infty}^{\infty} \frac{[p'(x)]^2}{p(x)} dx < \infty, \tag{1}$$

where $p(x)$ is the pdf of η_l, assumed to be continuously differentiable, and with support $(-\infty, \infty)$. Note that $I(\eta)$ is the Fisher information in x_l about θ, and is a deterministic value which does not depend on θ. In the following, η denotes a random variable with the same distribution as any η_l.

We present the following theorem, which provides two bounds involving $I(\eta)$ and $\varphi(\omega)$. It was proved first in [4] using the Cramér-Rao inequality. We provide an alternate proof which also delineates the condition for equality for the first time in the literature. The condition for equality will be central in Section III to establish necessary and sufficient conditions for the asymptotic efficiency of a distributed estimation algorithm over a Gaussian multiple-access channel.

Theorem 1: Let $\varphi_R(\omega)$ and $\varphi_I(\omega)$ be the real and the imaginary parts of $\varphi(\omega)$, respectively. We have

$$\omega^2 \varphi_I^2(\omega) \leq I(\eta) \left[\frac{1}{2} \left[1 + \varphi_R(2\omega) \right] - \varphi_R^2(\omega) \right], \tag{2}$$

$$\omega^2 \varphi_R^2(\omega) \leq I(\eta) \left[\frac{1}{2} \left[1 - \varphi_R(2\omega) \right] - \varphi_I^2(\omega) \right], \tag{3}$$

with equality in both (2) and (3) if and only if $\omega = 0$.

Proof: Let $s(x) := p'(x)/p(x)$ be the score function, where we recall that $p(x)$ is the pdf of η_l. Let $g(x)$ be a differentiable function satisfying $\lim_{x \to \pm\infty} g(x)p(x) = 0$. Using Stein's identity [7, Lemma 1.18], we have

$$E\left[g(\eta) s(\eta) \right] = -E\left[g'(\eta) \right]. \tag{4}$$

Applying the Cauchy-Schwarz inequality yields

$$E^2[g'(\eta)] \leq I(\eta) E[g^2(\eta)], \tag{5}$$

with equality if and only if $s(x) = \alpha g(x)$ for some α and all x. By substituting $g_1(x) := \cos(\omega x) - \varphi_R(\omega)$ for $g(x)$ in (5), equation (2) is obtained. Similarly, $g_2(x) := \sin(\omega x) - \varphi_I(\omega)$ substituted for $g(x)$ yields equation (3).

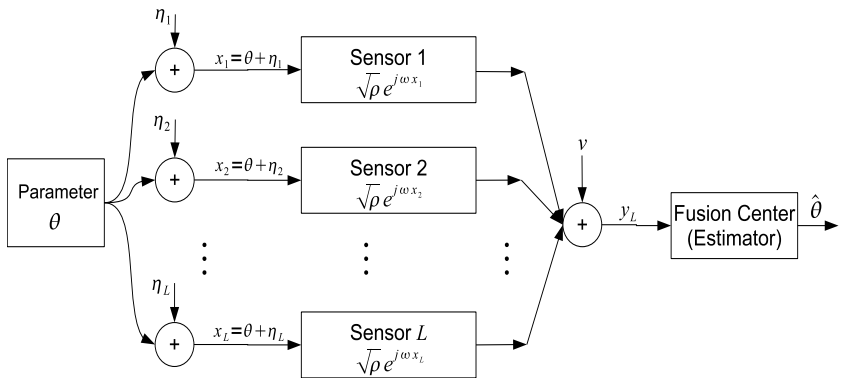

Fig. 1. System model: Wireless sensor network. The estimator is located at the fusion center.

To examine when equality occurs, first note that if $\omega = 0$, since $\varphi_R(0) = 1$ and $\varphi_I(0) = 0$, equations (2) and (3) become equalities. Conversely, consider $\omega \neq 0$. The equality condition for (3) is $s(x) = \alpha g_2(x)$, which yields the first order differential equation

$$\frac{p'(x)}{p(x)} = \alpha \left[\sin(\omega x) - \varphi_I(\omega)\right], \tag{6}$$

which must provide a solution satisfying $p(x) \geq 0$ and $\int_{-\infty}^{\infty} p(x)dx = 1$. The solution to (6) is of the form $p(x) = Ce^{-\alpha x \varphi_I(\omega)} e^{-\frac{\alpha}{\omega}\cos(\omega x)}$, which is unbounded as $x \to -\infty$ ($x \to \infty$) when $\varphi_I(\omega) > 0$ ($\varphi_I(\omega) < 0$), and periodic when $\varphi_I(\omega) = 0$. In either case, $\int_{-\infty}^{\infty} p(x)dx = 1$ is not possible. This shows that there is no pdf satisfying (6) when $\omega \neq 0$, and therefore, equality in (3) cannot be attained for $\omega \neq 0$. The same conclusion can be drawn about equation (2), using a similar argument with $s(x) = \alpha g_1(x)$. ■

III. Application to Distributed Estimation

A sensor network, illustrated in Figure 1, consisting of L sensors is considered. The value, x_l, observed at the lth sensor is

$$x_l = \theta + \eta_l \tag{7}$$

for $l = 1, ..., L$, where θ is a deterministic, real-valued, unknown parameter in a bounded interval of known length, $[0, \theta_R]$, where $\theta_R < \infty$, and η_l are iid real-valued random variables. We will assume that η_l has zero mean and variance σ_η^2, when the mean and variance exist. Due to constraints in the transmit power, we consider a scheme where the lth sensor transmits its

measurement, x_l, using a constant modulus base-band equivalent signal, $\sqrt{\rho}e^{j\omega x_l}$, over a Gaussian multiple access channel so that the received signal at the fusion center is given by

$$y_L = \sqrt{\rho}\sum_{l=1}^{L} e^{j\omega x_l} + \nu, \tag{8}$$

where the transmitted signal at each sensor has a per-sensor power of ρ, $\omega \in (0, 2\pi/\theta_R]$ is a design parameter to be optimized, and $\nu \sim \mathcal{CN}(0, \sigma_\nu^2)$ is independent of $\{\eta_l\}_{l=1}^L$. Note that the restriction $\omega \in (0, 2\pi/\theta_R]$ is necessary even in the absence of sensing and channel noise ($y_L = \sqrt{\rho}e^{j\omega\theta}$) to uniquely determine θ from y_L.

In a centralized problem, θ is estimated from $\{x_l\}_{l=1}^L$. The Fisher Information in $\{x_l\}_{l=1}^L$ about θ is given by $[LI(\eta)]$ [8, pp. 120], which makes the normalized per-sensor information, $I(\eta)$. For large L, the asymptotic variance is an appropriate performance metric, which is the variance of the asymptotic distribution of the normalized statistic $\sqrt{L}(\hat{\theta}_L - \theta)$, which is very often asymptotically normal, where $\hat{\theta}_L$ is the estimate of θ, and will be given in (10). Under certain regularity conditions, a lower bound on the asymptotic variance is given by $[I(\eta)]^{-1}$ [8, pp. 439]. Hence, the Fisher information has a central role to play in establishing benchmarks for the estimation of a location parameter for centralized estimation problems which address estimators of θ based on $\{x_l\}_{l=1}^L$.

For the distributed setting, based on (8), the estimators of θ are functions of the scalar, y_L. The desire to have constant modulus transmissions over a Gaussian multiple-access channel causes the fusion center in Figure 1 to have access to only y_L, rather than $\{x_l\}_{l=1}^L$. Clearly, y_L has less information about θ than $\{x_l\}_{l=1}^L$. In what follows, we quantify this loss by examining the efficiency of the minimum (asymptotic) variance estimator, and comparing it with the benchmark for the centralized problem, $[I(\eta)]^{-1}$, for different distributions on the sensing noise, η. Our main result in Theorem 2 uses Theorem 1 to show that there is no loss in efficiency if and only if η is Gaussian.

A. *The Estimator*

To estimate θ, we normalize y_L in (8) and define:

$$z_L := \frac{y_L}{L} = \sqrt{\rho}e^{j\omega\theta}\frac{1}{L}\sum_{l=1}^{L} e^{j\omega\eta_l} + \frac{\nu}{L}, \tag{9}$$

where $z_L = |z_L| \exp(j\angle z_L) = z_L^R + jz_L^I$, and z_L^R and z_L^I are the real and imaginary parts of z_L, respectively and $|z_L|$ and $\angle z_L$ are the magnitude and phase of z_L, respectively. The magnitude and phase are given by $|z_L| = \sqrt{(z_L^R)^2 + (z_L^I)^2}$ and $\angle z_L = \tan^{-1}(z_L^I/z_L^R)$, respectively, where the $\tan^{-1}(\cdot)$ function takes into account the quadrant in which z_L lies. Also define $\mathbf{z}_L := [z_L^R \ , \ z_L^I]^T$ and $\bar{\mathbf{z}}(\theta) := [E[z_L^R] \ , \ E[z_L^I]]^T = \sqrt{\rho}[\varphi_R(\omega)\cos\omega\theta - \varphi_I(\omega)\sin\omega\theta \ , \ \varphi_R(\omega)\sin\omega\theta + \varphi_I(\omega)\sin\omega\theta]^T$.

Given y_L (or equivalently $\mathbf{z}_L$), the estimator with the smallest asymptotic variance is given by [9, (3.6.2), pp. 82]

$$\hat{\theta}_L = \underset{\theta}{\operatorname{argmin}} [\mathbf{z}_L - \bar{\mathbf{z}}(\theta)] \mathbf{\Sigma}^{-1}(\theta) [\mathbf{z}_L - \bar{\mathbf{z}}(\theta)]^T, \tag{10}$$

where

$$\mathbf{\Sigma}(\theta) = \begin{bmatrix} \Sigma_{11}(\theta) & \Sigma_{12}(\theta) \\ \Sigma_{21}(\theta) & \Sigma_{22}(\theta) \end{bmatrix} \tag{11}$$

is the 2×2 asymptotic covariance matrix of $\mathbf{z}_L$, satisfying $\lim_{L\to\infty} \sqrt{L}[\mathbf{z}_L - \bar{\mathbf{z}}(\theta)] = \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}(\theta))$. Its elements are given by

$$\Sigma_{11}(\theta) = \rho \left[v_c \cos^2(\omega\theta) + v_s \sin^2(\omega\theta) \right]$$

$$\Sigma_{22}(\theta) = \rho \left[v_s \cos^2(\omega\theta) + v_c \sin^2(\omega\theta) \right]$$

$$\Sigma_{12}(\theta) = \Sigma_{21}(\theta) = \rho(v_c - v_s) \sin(\omega\theta) \cos(\omega\theta),$$

where $v_c := \text{var}[\cos(\omega\eta_l)] = 1/2 + \varphi_R(2\omega)/2 - \varphi_R^2(\omega)$ and $v_s := \text{var}[\sin(\omega\eta_l)] = 1/2 - \varphi_R(2\omega)/2 - \varphi_I^2(\omega)$.

Estimators of the form in (10) have an asymptotic variance given by [9, Lemma 3.1]

$$\text{AsV}(\omega) = \left[\left(\frac{\partial \bar{\mathbf{z}}(\theta)}{\partial \theta} \right)^T \mathbf{\Sigma}^{-1}(\theta) \left(\frac{\partial \bar{\mathbf{z}}(\theta)}{\partial \theta} \right) \right]^{-1}. \tag{12}$$

Substituting $\partial \bar{\mathbf{z}}(\theta)/\partial\theta = \sqrt{\rho}\omega[-\varphi_R(\omega)\sin\omega\theta - \varphi_I(\omega)\cos\omega\theta \ , \ \varphi_R(\omega)\cos\omega\theta - \varphi_I(\omega)\sin\omega\theta]^T$ and $\mathbf{\Sigma}^{-1}(\theta)$ whose elements can be expressed in terms of $\Sigma_{11}(\theta)$, $\Sigma_{22}(\theta)$ and $\Sigma_{12}(\theta)$, the asymptotic variance is given by

$$\begin{aligned} \text{AsV}(\omega) &= \frac{2v_c v_s}{\omega^2 \left[v_s \varphi_I^2(\omega) + v_c \varphi_R^2(\omega) \right]} \\ &= \frac{\left(1 + \varphi_R(2\omega) - 2\varphi_R^2(\omega)\right)\left(1 - \varphi_R(2\omega) - 2\varphi_I^2(\omega)\right)}{\omega^2 \left[\varphi_R^2(\omega)\left(1 + \varphi_R(2\omega) - 2\varphi_R^2(\omega)\right) + \varphi_I^2(\omega)\left(1 - \varphi_R(2\omega) - 2\varphi_I^2(\omega)\right) \right]}. \end{aligned} \tag{13}$$

Note that $\text{AsV}(\omega)$ depends on the sensing noise through its characteristic function, and does not depend on the channel noise variance, σ_ν^2, which washes out for large L.

B. Asymptotic Efficiency

We now address the asymptotic efficiency of $\hat{\theta}_L$ and characterize the condition under which $\mathrm{AsV}(\omega)$ can be made arbitrarily close to $[I(\eta)]^{-1}$:

Theorem 2: The estimator in (10) can be arbitrarily close to being asymptotically efficient by the proper choice of ω, that is,

$$\inf_{\omega\in(0,2\pi/\theta_R]} \mathrm{AsV}(\omega) = \frac{1}{I(\eta)}, \tag{14}$$

if and only if η is Gaussian.

Proof: We begin by showing that if (14) holds, then η is Gaussian. Using Theorem 1, the inequalities in (2) and (3) can be rewritten for $\omega > 0$ as

$$\frac{\omega^2\varphi_I^2(\omega)}{\frac{1}{2}\left[1+\varphi_R(2\omega)\right]-\varphi_R^2(\omega)} < I(\eta), \tag{15}$$

$$\frac{\omega^2\varphi_R^2(\omega)}{\frac{1}{2}\left[1-\varphi_R(2\omega)\right]-\varphi_I^2(\omega)} < I(\eta), \tag{16}$$

where we use that when $\omega \neq 0$, (2) and (3) are strict inequalities. Adding the inequalities in (15) and (16), rearranging the resulting inequality and recalling (13), we have

$$\frac{1}{I(\eta)} < \mathrm{AsV}(\omega), \qquad \omega \in (0, 2\pi/\theta_R]. \tag{17}$$

Equation (17) indicates that the infimum in (14) is not attained for any non-zero finite value of ω. Since ω is bounded above, the only way for (14) to hold is when $\lim_{\omega\to 0}\mathrm{AsV}(\omega) = [I(\eta)]^{-1}$. It is easy to verify, using L'Hospital's rule, that $\lim_{\omega\to 0}\mathrm{AsV}(\omega) = \sigma_\eta^2$, the variance of η_l. Therefore, for (14) to hold, we have $[I(\eta)]^{-1} = \sigma_\eta^2$. The only distribution that satisfies this is the Gaussian [7, Lemma 1.19]. This completes the proof of the first half.

To show that (14) holds when η_l is Gaussian, $\varphi(\omega) = e^{-\omega^2\sigma_\eta^2/2}$ is substituted into (13) to yield:

$$\mathrm{AsV}(\omega) = \frac{1}{\omega^2}e^{-\sigma_\eta^2\omega^2}\left(e^{2\sigma_\eta^2\omega^2}-1\right)^2. \tag{18}$$

Since $\mathrm{AsV}(\omega)$ is continuous and $\lim_{\omega\to 0}\mathrm{AsV}(\omega) = [I(\eta)]^{-1}$, the proof will be furnished if we show that $\mathrm{AsV}(\omega) < [I(\eta)]^{-1}$ is not possible. But, (18) is non-decreasing in ω, since

$$\frac{\partial \mathrm{AsV}(\omega)}{\partial\omega} = \frac{2e^{-2\sigma_\eta^2\omega^2}}{\omega^3}\left(e^{2\sigma_\eta^2\omega^2}-1\right)\left[(1-e^{2\sigma_\eta^2\omega^2})+2\sigma_\eta^2\omega^2+2\sigma_\eta^2\omega^2e^{2\sigma_\eta^2\omega^2}\right] \geq 0, \tag{19}$$

for $\omega > 0$. ■

The phase modulation scheme considered here has the advantage of constant modulus transmissions. Due to the use of phase modulation, the result in Theorem 2 is related to the efficiency of the estimator of a location parameter using the empirical characteristic function (ECF), defined as $\hat{\varphi}(\omega) := L^{-1}\sum_{l=1}^{L} e^{j\omega x_l}$. It can be seen from (9) that $z_L = \sqrt{\rho}e^{j\omega\theta}\hat{\varphi}(\omega)+\nu/L$ is related to the ECF through scaling and additive noise. The efficiency of empirical characteristic function based estimators has been considered for arbitrary parameters (that is, not just location parameters) in [10], but with a continuum of infinitely many values of the argument, ω, of the ECF. In the current distributed estimation application, the evaluation of $\hat{\varphi}(\omega)$ for many values of ω at the fusion center corresponds to many transmissions per sensor observation, requiring large bandwidth. In contrast, we consider a single value of ω for estimation, requiring a single transmission per sensor. The analog transmissions are assumed to be appropriately pulse-shaped and phase modulated to consume finite bandwidth.

When the sensing noise distribution is symmetric, the cost function on the right hand side of (10) that needs to be minimized can be expressed as

$$\begin{aligned} c(\theta) =& [\mathbf{z}_L - \bar{\mathbf{z}}(\theta)]\mathbf{\Sigma}^{-1}(\theta)[\mathbf{z}_L - \bar{\mathbf{z}}(\theta)]^T \\ =& \frac{1}{2\rho^2 v_c v_s}\Big[-4\rho^{3/2} v_s \varphi(\omega)[z_L^I \sin(\omega\theta) + z_L^R \cos(\omega\theta)] + 2\rho^2 v_s \varphi^2(\omega) \\ &+ \rho(v_c - v_s)\left((z_L^I)^2 - (z_L^R)^2\right)\cos(2\omega\theta) - 2\rho(v_c+v_s) z_L^I z_L^R \sin(2\omega\theta) \\ &+ \rho(v_c+v_s)\left((z_L^I)^2 + (z_L^R)^2\right)\Big]. \end{aligned} \tag{20}$$

Differentiating with respect to θ, we have

$$\begin{aligned} \frac{\partial c(\theta)}{\partial\theta} =& \frac{2\omega z_L^R \cos(\omega\theta)}{\rho v_c v_s}\left[\frac{z_L^I}{z_L^R} - \tan(\omega\theta)\right] \\ &\times \left[\left(1 + \frac{z_L^I}{z_L^R}\tan(\omega\theta)\right) v_c + \left(1 - \frac{\sqrt{\rho}\varphi(\omega)}{z_L^R\cos(\omega\theta)}\frac{z_L^I}{z_L^R}\tan(\omega\theta)\right) v_s\right]. \end{aligned} \tag{21}$$

The values of θ at which (21) is zero are given by

$$\theta \in \left\{ \frac{n\pi \pm \frac{\pi}{2}}{\omega}, \frac{1}{\omega}\angle z_L, \frac{\angle z_L + 2n\pi \pm \frac{\pi}{2}}{\omega} \right\}, \tag{22}$$

where $\omega \neq 0$ and $n \in \mathbb{Z}^+$. The value of θ that minimizes $c(\theta)$ is easily verified by substituting the values of θ from (22) into (20) and is given by

$$\hat{\theta} = \frac{1}{\omega}\angle z_L. \tag{23}$$

Hence, in the presence of symmetric noise, the estimator in (10) that minimizes the asymptotic variance reduces to the simple expression in (23), which was first considered in [3], [11]. However, in [3], [11], neither the optimality (in terms of minimizing the asymptotic variance) nor the efficiency of the estimator in (23) was considered.

C. Quantifying Relative Efficiency

One way of interpreting Theorem 2 is to observe that when the sensing noise is Gaussian, no information is lost by analog phase modulation if ω is chosen sufficiently small. On the other hand, information is lost when the sensing noise follows any other distribution. To see this more clearly, we define the relative efficiency between the asymptotic variance and the Fisher information as:

$$\mathcal{E}(\eta) = \left[I(\eta) \inf_{\omega \in (0, 2\pi/\theta_R]} \mathrm{AsV}(\omega) \right]^{-1}. \tag{24}$$

It can easily be verified that $\mathcal{E}(\eta)$ is scale-invariant in the sense that $\mathcal{E}(\alpha\eta) = \mathcal{E}(\eta)$ for any $\alpha \in \mathbb{R}$. Moreover, based on Theorem 2 and (17), $0 \leq \mathcal{E}(\eta) \leq 1$, where the equality in the upper-bound is achieved only if η is Gaussian.

The relative efficiency in (24) depends only on the distribution of the sensing noise. The values of $\mathcal{E}(\eta)$ for several distributions are provided in Table I. The result in Table I for the Gaussian case has been established in Theorem 2. For the Laplace sensing noise, $\varphi(\omega) = (1 + \omega^2\sigma_\eta^2/2)^{-1}$, $\mathrm{AsV}(\omega) = \sigma_\eta^2(1 + \sigma_\eta^2\omega^2/2)/(1 + 2\sigma_\eta^2\omega^2)$, and $\inf_{\omega \in (0, 2\pi/\theta_R]} \mathrm{AsV}(\omega) = 3\sigma_\eta^2/4$, by inspecting the third derivative of $\mathrm{AsV}(\omega)$. Similarly for the Cauchy case, $\varphi(\omega) = e^{-\gamma\omega}$, $\mathrm{AsV}(\omega) = e^{2\gamma\omega}(1 - e^{-2\gamma\omega})/2\omega^2$, and $\inf_{\omega \in (0, 2\pi/\theta_R]} \mathrm{AsV}(\omega) = 4\gamma^2 e^c(1 - e^{-2c})/c^2$, by examining the first derivative of $\mathrm{AsV}(\omega)$ where γ is the scale parameter of the Cauchy random variable, $c := 2 + W(-2e^{-2})$, and $W(\cdot)$ is the Lambert W-function [12]. For the uniform distribution, an extension of the definition in (1) can be used [13, §4.4]:

$$I(\eta) = \sup_{\psi(x)} \frac{\left[\int \psi'(x) dF(x)\right]^2}{\int \psi^2(x) dF(x)}, \tag{25}$$

where the supremum is taken over continuously differentiable functions with compact support, satisfying $\int \psi^2(x) dF(x) > 0$ and $F(x)$ is the cdf of η. This definition can be used to argue that in the case of the uniform distribution, the Fisher information is infinite [8, pp. 119], and the relative efficiency of the estimator as defined in (24) is zero.

Distribution	Gaussian	Laplace	Cauchy	Uniform
$\mathcal{E}(\eta)$	1	$2/3$	$0.5c^2e^{-c}(1-e^{-c})^{-1} \approx 0.65$	0

TABLE I

$\mathcal{E}(\eta)$ FOR DIFFERENT DISTRIBUTIONS.

We have seen that the Gaussian sensing noise is the only distribution with the highest possible efficiency when the observations x_l are transmitted with phase modulation over Gaussian multiple-access channels and the estimator in (10) is used. However, it is possible that other sensing noise distributions, which yield less efficiency, have better asymptotic variances. This is because efficiency is defined relative to the Fisher information. For example, for Laplace sensing noise, the proposed estimator is not asymptotically efficient, but has better asymptotic variance than in the Gaussian case, since its inverse Fisher information, $[I(\eta)]^{-1}$, is lower. In conclusion, Gaussian sensing noise has the only distribution that does not suffer a loss in efficiency when the sensed data x_l is mapped to constant modulus transmissions over Gaussian multiple-access channels.

IV. Numerical Results

In Figures 2 and 3, the asymptotic variance and the value of $[I(\eta)]^{-1}$ in dB are plotted versus ω, when the sensing noise is Gaussian, Laplace, uniform and Cauchy distributed.

From Figure 2, it can be seen that the asymptotic variance approaches $[I(\eta)]^{-1}$ only as $\omega \to 0$ for Gaussian sensing noise, and is bounded away from $[I(\eta)]^{-1}$ for other values of ω. The estimator in (10) is not efficient when the sensing noise is non-Gaussian. Using the definition of relative efficiency in (24), it can be seen from Figure 2 that $\mathcal{E}(\eta)$ in the case of Gaussian sensing noise is 0dB, and in the case of Laplace sensing noise is about -3.5dB. In Figure 2, it can be verified that $\inf_\omega \text{AsV}(\omega) \approx 0.75$, which is about -2.5dB at $\omega = 1/\sqrt{2}$, which is lower than the Gaussian sensing noise case.

From Figure 3 the relative efficiency for the Cauchy noise case is about -3.8dB, verifying the value shown in Table I. The inverse Fisher information for the uniform case is zero ($-\infty$ dB) and is not shown in Figure 3. The relative efficiency as defined in (24), for uniform noise, is therefore zero. When the sensing noise follows the Cauchy, uniform or Laplace distributions,

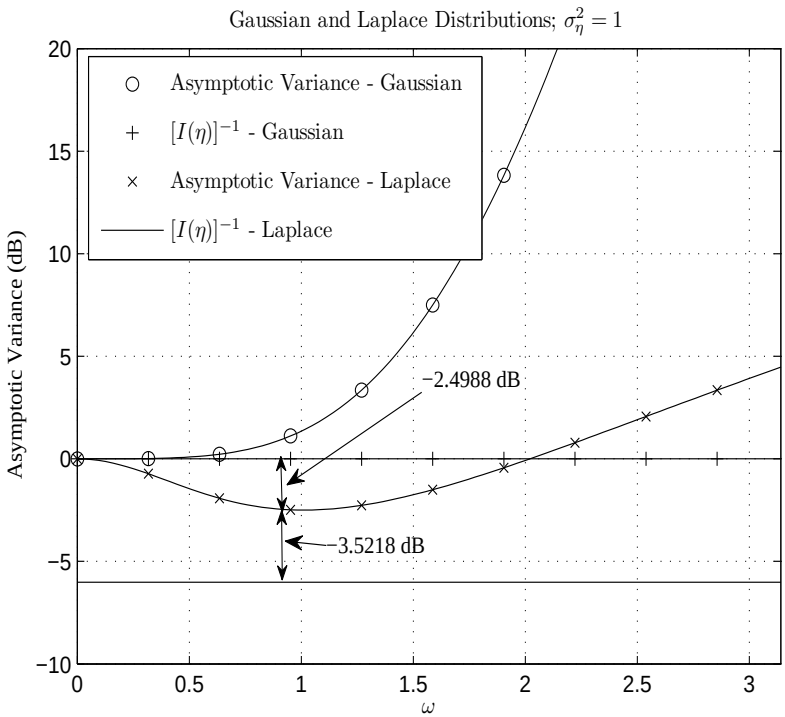

Fig. 2. Plot of asymptotic variance vs. ω.

the estimator is not asymptotically efficient.

V. Conclusions

In this paper, we considered a distributed estimation problem with a Gaussian multiple-access channel between the sensors and the FC. Each sensor phase-modulates its observation for transmission to the FC. The location parameter of the sensed data is estimated at the FC. While our design is robust to several noise distributions, the asymptotic efficiency of the estimator is calculated for sensing noise following the Gaussian, Laplace, Cauchy and uniform distributions. The relationship between the Fisher information and the characteristic function is investigated through two bounds. The condition for equality is also derived, for the first time in literature, and used to show that the estimator is asymptotically efficient if and only if the sensing noise is Gaussian distributed.

Our primary interests are in (i) designing the sensor mapping, which in this case is $x_l \to e^{j\omega x_l}$; and (ii) designing the estimator $\hat{\theta}_L$. By comparing the normalized information that $\{x_l\}_{l=1}^L$ has

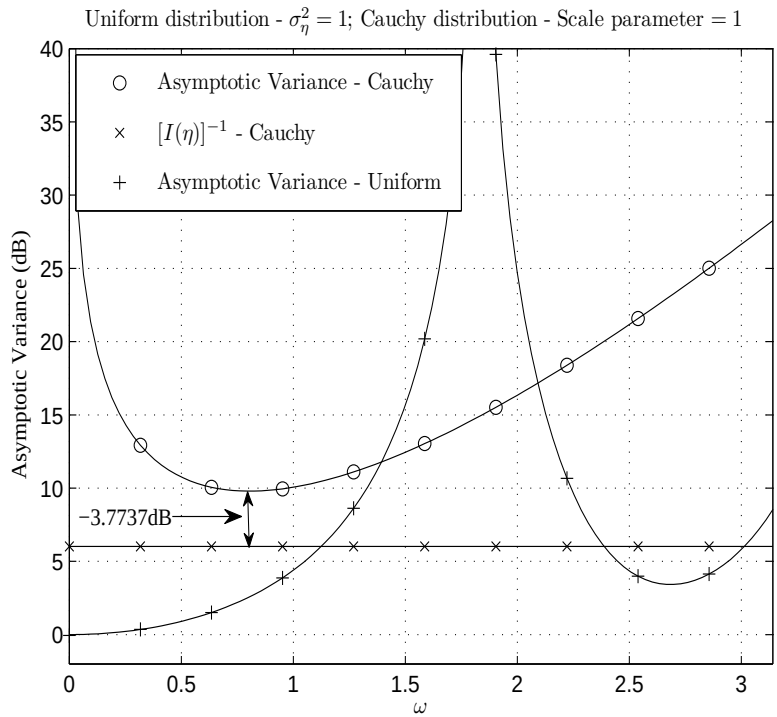

Fig. 3. Plot of asymptotic variance vs. ω. Note that the value of $[I(\eta)]^{-1}$ is 0 ($-\infty$ dB) for the uniform sensing noise case and is not shown.

about θ with the asymptotic variance, we are jointly assessing the merits of both the sensor design as well as the estimator. Therefore, the efficiency as defined in (24) reflects both the sensor mapping and the estimator at the FC. As mentioned, the efficiency result was obtained using the relationship between the Fisher information and the characteristic function. In all cases, the loss in efficiency was quantified through a scale-invariant relative efficiency metric that takes values between 0 and 1. This metric depends only on the distribution of the sensing noise used, and was computed for the Gaussian, Laplace, Cauchy, and uniform sensing distributions. These relative efficiency values can be interpreted as the amount of information lost due to constant modulus transmissions over Gaussian multiple-access channels relative to having perfect access to all sensor measurements. The Gaussian sensing noise is the only one not suffering from such an information loss, as also verified in our numerical results.

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