Solving nonlinear PDEs using RBF-FD

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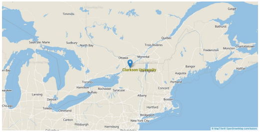


Nonlinear time-dependent PDEs

Consider the following general form of nonlinear time-dependent PDE problem:

$$\begin{cases} \dfrac{\partial u(\mathbf{x},t)}{\partial t} = \mathcal{D}u(\mathbf{x},t) + f(\mathbf{x},t,u(\mathbf{x},t)), & \mathbf{x} \in \Omega \subset \mathbb{R}^d, \\ \mathcal{B}u(\mathbf{x},t) = g(\mathbf{x},t,u(\mathbf{x},t)), & \mathbf{x} \in \partial\Omega, \\ u(\mathbf{x},0) = u^0(\mathbf{x}), & \mathbf{x} \in \Omega \cup \partial\Omega, \end{cases} \tag{1}$$

where $\mathcal{D}$ is a nonlinear partial differential operator and $\mathcal{B}$ is the boundary differential operator, f, g, and u^0 are known linear or nonlinear functions, Ω is a computational domain, and $\partial\Omega$ is the boundary of Ω.

Nonlinear time-dependent PDEs (Cont'd)

FDM in time:

- Time-space is discretized by the Implicit Time Stepping method:

$$\frac{\partial u(\mathbf{x},t)}{\partial t} \approx \frac{u(\mathbf{x},t) - u(\mathbf{x},t_0)}{h} = \mathcal{D}u(\mathbf{x},t) + f(\mathbf{x},t,u(\mathbf{x},t)). \quad (2)$$

where $t = t_0 + h$ and h is the time step size.

Rearranging (2) and rewriting the boundary condition at t, gives

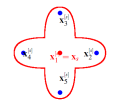
$$\begin{cases} h\mathcal{D}u(\mathbf{x},t) - u(\mathbf{x},t) + u(\mathbf{x},t_0) + hf(\mathbf{x},t,u(\mathbf{x},t)) = 0, & \mathbf{x} \in \Omega, \\ \mathcal{B}u(\mathbf{x},t) - g(\mathbf{x},t,u(\mathbf{x},t)) = 0, & \mathbf{x} \in \partial\Omega. \end{cases} \quad (3)$$

This is a nonlinear elliptic PDE, where $u(\mathbf{x},t)$ is the unknown function. We will use the localized implicit method of approximated particular solutions (LMAPS) to find it. It actually is almost the same as RBF-FD.

Nonlinear time-dependent PDEs (Cont'd)

General elliptic equations:

$$\begin{cases} \tilde{\mathcal{D}}u(\mathbf{x}) &= f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ \tilde{\mathcal{B}}u(\mathbf{x}) &= g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega \end{cases} \tag{4}$$



- Spatial-space is discretized by the LMAPS (RBF-FD) using polyharmonic splines together with the polynomial bases.

- Collocation technique:

$$\begin{bmatrix} \hat{u}(\mathbf{x}_1^{[s]}) \\ \hat{u}(\mathbf{x}_2^{[s]}) \\ \vdots \\ \hat{u}(\mathbf{x}_n^{[s]}) \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi(\|\mathbf{x}_1^{[s]} - \mathbf{x}_1^{[s]}\|) & \Phi(\|\mathbf{x}_1^{[s]} - \mathbf{x}_2^{[s]}\|) & \cdots & \Phi(\|\mathbf{x}_1^{[s]} - \mathbf{x}_n^{[s]}\|) & 1 & x_1 & y_1 \\ \Phi(\|\mathbf{x}_2^{[s]} - \mathbf{x}_1^{[s]}\|) & \Phi(\|\mathbf{x}_2^{[s]} - \mathbf{x}_2^{[s]}\|) & \cdots & \Phi(\|\mathbf{x}_2^{[s]} - \mathbf{x}_n^{[s]}\|) & 1 & x_2 & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi(\|\mathbf{x}_n^{[s]} - \mathbf{x}_1^{[s]}\|) & \Phi(\|\mathbf{x}_n^{[s]} - \mathbf{x}_2^{[s]}\|) & \cdots & \Phi(\|\mathbf{x}_n^{[s]} - \mathbf{x}_n^{[s]}\|) & 1 & x_n & y_n \\ 1 & 1 & \cdots & 1 & 0 & 0 & 0 \\ x_1 & x_2 & \cdots & x_n & 0 & 0 & 0 \\ y_1 & y_2 & \cdots & y_n & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1^{[s]} \\ \alpha_2^{[s]} \\ \vdots \\ \alpha_n^{[s]} \\ \alpha_{n+1}^{[s]} \\ \alpha_{n+2}^{[s]} \\ \alpha_{n+3}^{[s]} \end{bmatrix}.$$

History of LMAPS, really it is just RBF-FD

$$\tilde{\mathcal{D}}u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$
$$\tilde{\mathcal{B}}u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega$$

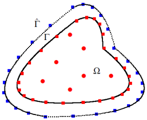
Boundary Value Problem (BVP)

- **Kansa's Method**

$$\hat{u}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \phi(\|\mathbf{x} - \mathbf{x}_j\|)$$

- **Method of Fundamental Solutions**

$$\hat{u}(\mathbf{x}) = \sum_{j=1}^{m} \alpha_j G(\mathbf{x}, \mathbf{x}_j^s)$$



$$\tilde{\mathcal{D}}G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x},\mathbf{y})$$

$$\tilde{\mathcal{D}}u(\mathbf{x}) = 0,$$
$$\tilde{\mathcal{B}}u(\mathbf{x}) = g(\mathbf{x}$$

- **Two-stage MFS-MPS**

$$u = u_h + u_p$$

- **One-stage MFS-MPS**

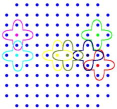
$$\hat{u}(\mathbf{x}) = \sum_{j=1}^{N_i} \alpha_j \Phi(\|\mathbf{x} - \mathbf{x}_j\|) + \sum_{l=1}^{N_b} \beta_l G(\|\mathbf{x} - \mathbf{x}_l^s\|)$$

- **Method of Particular Solutions (MPS)**

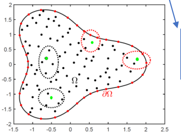
$$\hat{u}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \Phi(\|\mathbf{x} - \mathbf{x}_j\|) \qquad \tilde{\mathcal{D}}\Phi = \phi$$

- **Localized MPS (LMPS or LMAPS)**

$$u(\mathbf{x}_s) \approx \hat{u}(\mathbf{x}_s) = \sum_{k=1}^{n} \alpha_k^{[s]} \Phi(\|\mathbf{x}_s - \mathbf{x}_k^{[s]}\|)$$



$$\begin{bmatrix} \Xi(\mathbf{x}_1) \\ \vdots \\ \Xi(\mathbf{x}_{N_i}) \\ \Upsilon(\mathbf{x}_{N_i+1}) \\ \vdots \\ \Upsilon(\mathbf{x}_N) \end{bmatrix} \begin{bmatrix} \hat{u}(\mathbf{x}_1) \\ \vdots \\ \hat{u}(\mathbf{x}_{N_i}) \\ \hat{u}(\mathbf{x}_{N_i+1}) \\ \vdots \\ \hat{u}(\mathbf{x}_N) \end{bmatrix} = \begin{bmatrix} f(\mathbf{x}_1) \\ \vdots \\ f(\mathbf{x}_{N_i}) \\ g(\mathbf{x}_{N_i+1}) \\ \vdots \\ g(\mathbf{x}_N) \end{bmatrix}$$

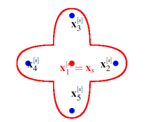


LMAPS for Elliptic Equations

General Elliptic Equations:

$$\tilde{\mathcal{D}}u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \qquad (5)$$

$$\tilde{\mathcal{B}}u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega \qquad (6)$$



LMAPS using PH of order k and polynomials of order m:

$$u(\mathbf{x}_i) \approx \hat{u}(\mathbf{x}_i) = \sum_{j=1}^{n} \alpha_j \Phi(||\mathbf{x}_i - \mathbf{x}_j^{[i]}||) + \sum_{l=1}^{w} \alpha_{n+l} p_l(\mathbf{x}_i); \quad \mathbf{x}_i \in \Omega. \qquad (7)$$

Collocation in local domains: $u(\mathbf{x}) = \Phi\alpha \Rightarrow \mathbf{u} = \Phi\alpha \Rightarrow \alpha = \Phi^{-1}\mathbf{u}$

Differentiation of unknowns:

$$\tilde{\mathcal{D}}u(\mathbf{x}) = \tilde{\mathcal{D}}\Phi(\mathbf{x})\Phi^{-1}\mathbf{u} \qquad (8)$$

$$\tilde{\mathcal{B}}u(\mathbf{x}) = \tilde{\mathcal{B}}\Phi(\mathbf{x})\Phi^{-1}\mathbf{u} \qquad (9)$$

LMAPS for Elliptic Equations

Discretized linear elliptic equations in matrix-vector form:

$$A\mathbf{u} = \mathbf{b}, \tag{10}$$

where A is a global sparse matrix, $\mathbf{u}$ is the unknown solution of the given elliptic equation at a set of evaluation points of interests. This can be solved by a sparse system solver. Note that this is similar to what so called RBF-FD scheme.

Discretized nonlinear elliptic equations in matrix-vector form:

$$A(\mathbf{u}) = \mathbf{b}. \tag{11}$$

This is a system of nonlinear algebraic equations, which can be solved by nonlinear solver for system of equations, such as Picard method or Newton-Raphson method.

Example 1 – Interpolation of Franke's Functions

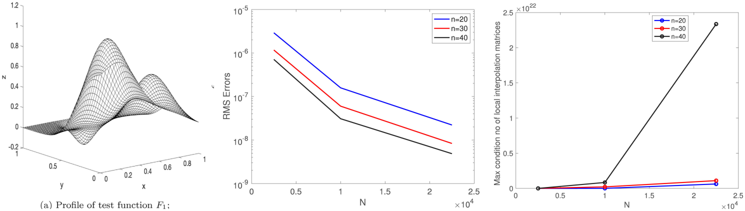


Figure 1: Left: F1; Middle: RMS errors vs total number of interpolation points; Right: maximum condition number of local matrices with order of PH $k = 4$, and order of polynomials $m = 3$.

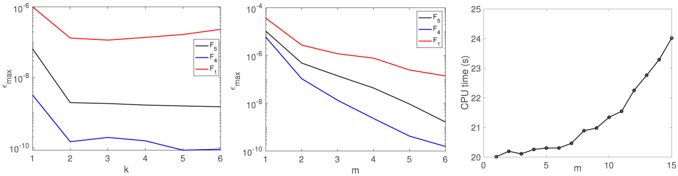


Figure 2: $N = 100^2$, $N_t = 9000$, $n = 30$. Left: $m = 6$; Middle: $k = 4$; Right: CPU time.

Example 2 – Nonlinear Elliptic

Nonlinear elliptic equation with mixed BCs:

$$\Delta u(x,y) + y\cos(y)\frac{\partial u(x,y)}{\partial x} - x\sin(x)\frac{\partial u(x,y)}{\partial y} + u^2(x,y) = f(x,y)$$

$$u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega^D,$$

$$\frac{\partial u(x,y)}{\partial \mathbf{n}} = h(x,y), \quad (x,y) \in \partial\Omega^N,$$

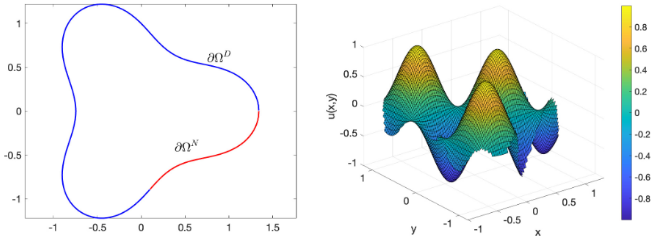


Figure 3: Left: domain; Right: analytical solution.

Example 2 – Nonlinear Elliptic (Cont'd)

Table 5. Example 4.4: Comparison of ϵ_{rms}, ϵ_∞ using ILMAPS with PHS and LMAPS with PBF for different order of polynomial basis with $n_i = 14{,}350$, $n_b = 400$.

m	LMAPS with PHS		LMAPS with PBF [Dangal *et al.* (2021)]	
	ϵ_{rms}	ϵ_∞	ϵ_{rms}	ϵ_∞
3	9.915E-07	2.406E-06	2.96E-03	2.73E-02
4	8.324E-08	2.785E-07	5.34E-04	2.91E-03
5	6.426E-08	1.670E-07	7.70E-06	6.19E-05
6	2.605E-09	3.924E-08	5.77E-06	5.53E-05

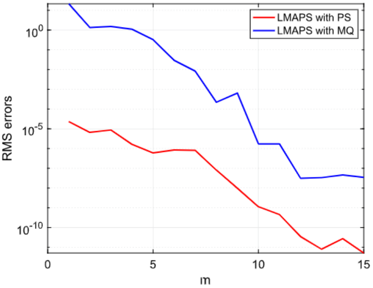
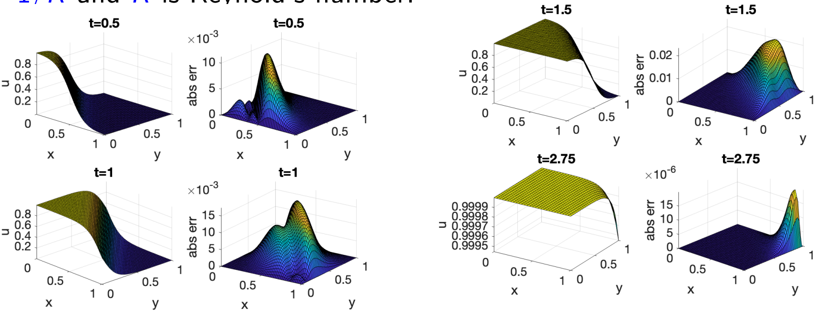


Figure 4: $N_i = 6830$, $N_b = 400$, $n = 150$, $k = 4$.

Example 3 – Nonlinear Burger's equation:

$$\frac{\partial u}{\partial t} + uu_x + uu_y = \alpha \Delta u, \qquad (x, y) \in \Omega, \tag{12}$$

where $\Omega = [0, 1]^2$ and initial and Dirichlet boundary conditions are obtained using the exact solution $u(x, y, t) = 1/(1 + e^{(x+y-t)/2\alpha})$, where $\alpha = 1/R$ and R is Reynold's number.



Profiles of the analytical solutions and the absolute errors at $t = 0.5, 1, 1.5,$and 2.75 computed with $k = 2, m = 2,$ and $n = 9$ and $R = 10$.

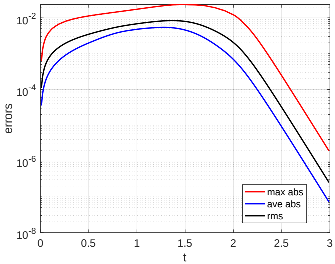


Figure 5: The errors as functions of time when 31×31uniformly distributed nodes are used. Note that $\Delta t = 0.01$, $k = 2$, $m = 2$ and $n = 9$.

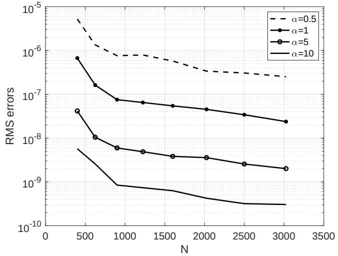


Figure 6: The rate of convergence with respect to N for various α when time $t = 3$.

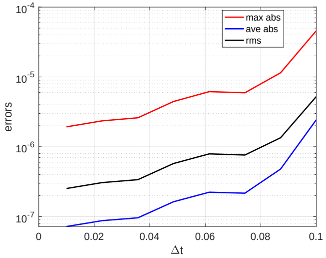
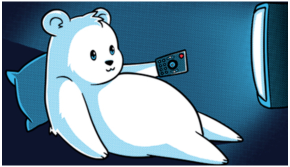


Figure 7: The rate of convergence with respect to Δt when time $t = 3$.

Conclusion

- The localized interpolation methods based on polyharmonic splines have been found to be highly accurate.
- High efficiency of ILMAPS when coupled with the Picard method for solving nonlinear elliptic and parabolic PDEs in both 2D and 3D.
- The method's benefits include using shape parameter-free polyharmonic splines and can improve accuracy through more points in local domains and higher-order polynomial basis functions combined with PHS.
- Ill-conditioning problem with small interpolation matrices – pre-conditioning needed?
- Me not be too lazy – work on RBF+nonlinear solver with high Reynold's number.



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Thank You!

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