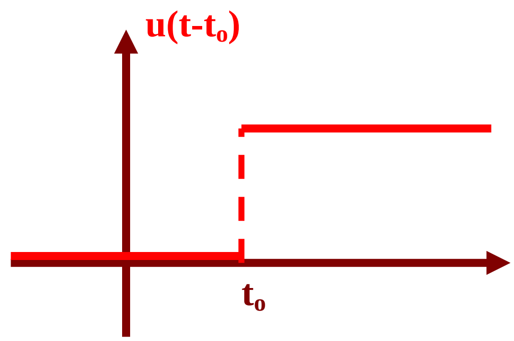
Review of Engineering Mathematics

Special Functions

Unit step function

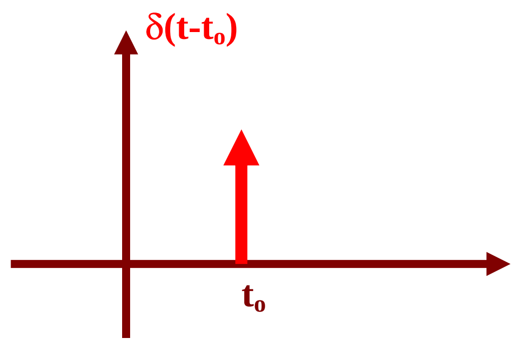
$$u(t-t_0)=\begin{Bmatrix}1 & t \ge t_0 \\ 0 & t < t_0\end{Bmatrix}$$



Delta Function

Dirac delta function is defined as

$$\delta(t-t_0)=\frac{du(t-t_0)}{dt}$$



Note that

$$\int_{-\infty}^{+\infty}\delta(t-t_0)dt=\int_{t_0-\varepsilon}^{t_0+\varepsilon}\delta(t-t_0)dt=1$$

Also

$$\int_{-\infty}^{+\infty}f(t)\delta(t-t_0)dt=\int_{t_0-\varepsilon}^{t_0+\varepsilon}f(t)\delta(t-t_0)dt=f(t_0)$$

$$\int_{-\infty}^{t}f(t_1)\delta(t_1-t_0)dt_1=f(t_0)u(t-t_0)$$

$$\delta[a(t-t_0)] = \frac{1}{|a|}\delta(t-t_0) \qquad \text{for} \qquad a \neq 0$$

Error Function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt\,, \quad \operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

$$\operatorname{erf}(0) = 0 = \operatorname{erfc}(\infty), \quad \operatorname{erf}(-x) = -\operatorname{erf}(x)$$

Exponential Integrals

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t}dt\,, \qquad E_i(x) = \int_{-\infty}^x \frac{e^t}{t}dt$$

Differential Equations

Linear First-Order Differential Equations

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y = ce^{-\int_0^x P(x_1)dx} + \int_0^x e^{-\int_{x_1}^x P(x_1)dx_2} Q(x_1)dx_1$$

Example

$$\frac{dy}{dx} + b\,y = Q(x), \qquad \text{with} \qquad y(0) = 0$$

$$y = \int_0^x e^{-b(x-x_1)}Q(x_1)dx_1$$

Second-Order Differential Equations with Constant Coefficients

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

Form characteristic equation

$m^2 + am + b = 0$ Solve for $\rightarrow$ m_1, m_2

Different cases:

i. m_1, m_2 are real and $m_2 \neq m_1$, then

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

ii. $m_2 = m_1 = m$ ~ real, then

$$y = e^{mx}(c_1 + c_2 x)$$

iii. $m_1 = p + qi$, $m_2 = p - qi$, where $p = -\frac{a}{2}$, $q = \sqrt{b - \frac{a^2}{4}}$, then

$$y = e^{px}\left(c_1 \cos qx + c_2 \sin qx\right)$$

Nonhomogeneous Second-Order Differential Equations (Particular Solutions)

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = R(x)$$

i. $$y_P = \frac{e^{m_1 x}}{m_1 - m_2}\int e^{-m_1 x} R(x)dx + \frac{e^{m_2 x}}{m_2 - m_1}\int e^{-m_2 x} R(x)dx$$

ii. $$y_P = xe^{mx}\int e^{-mx} R(x)dx - e^{mx}\int xe^{-mx} R(x)dx$$

iii. $$y_P = \frac{e^{px}\sin qx}{q}\int e^{-px} R(x)\cos qx\, dx - \frac{e^{px}\cos qx}{q}\int e^{-px} R(x)\sin qx\, dx$$

Bernoulli's Differential Equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Let

$$v = y^{1-n}$$

$$v = \frac{(1-n)\int Qe^{(1-n)\int P dx}dx + c}{e^{(1-n)\int P dx}}$$

Euler Differential Equation

$$x^2 \frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = s(x)$$

Change variable $x = e^t$. After some algebra we find

$$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = s(e^t)$$

which is a constant coefficient equation.

An alternative method that is more convenient is to assume a power law solution. That is

$$y = Ax^m$$

which leads to a characteristic equation given as

$$m(m-1) + am + b = 0$$

Solving for $m = m_1, m_2$. Then

$$y = A_1 x^{m_1} + A_2 x^{m_2}$$

Homogeneous Differential Equations

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Set $v = \frac{y}{x}$, then

$$\ln x = \int \frac{dv}{F(v) - v} + c$$

Exact Differential Equations

$$M(x,y)dx + N(x,y)dy = 0$$

with

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 \varphi(x,y)}{\partial x \partial y}$$

i.e.

$$M = \frac{\partial \varphi}{\partial x}, \qquad N = \frac{\partial \varphi}{\partial y}.$$

The solution then is given by

$$\varphi(x,y) = \text{const}$$

Riccati Differential Equation

$$\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$$

Let

$$y = -\frac{1}{A(x)v}\frac{dv}{dx}$$

Then

$$\frac{d^2v}{dx^2} - \left[\frac{1}{A}\frac{dA}{dx} + B(x)\right]\frac{dv}{dx} + A(x)C(x)v = 0,$$

which is a linear equation.

If a solution y_0 of the Riccati Equation is known, then the transformation

$$y = y_0(x) + u$$

leads to a Bernoulli's Equation. i.e.,

$$\frac{du}{dt} = (B + 2Ay_0)u + Au^2$$

Bessel's Differential Equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\beta^2 x^2 - n^2)y = 0$$

$$y = C_1 J_n(\beta x) + C_2 Y_n(\beta x)$$

where $J_n(\beta x)$, $Y_n(\beta x)$ are the Bessel functions of the first and second kinds, respectively.

Generalized Bessel's Equation

$$\frac{d^2y}{dx^2} + \frac{1-2a}{x}\frac{dy}{dx} + \left[(bcx^{c-1})^2 + \frac{a^2 - m^2c^2}{x^2}\right]y = 0$$

$$y = x^a\left(c_1 J_m(bx^c) + c_2 Y_m(bx^c)\right)$$

Modified Bessel's Differential Equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (\beta^2 x^2 + n^2)y = 0$$

$$y = c_1 I_n(\beta x) + c_2 K_n(\beta x)$$

where I_n and K_n are the modified Bessel functions.

Legendre's Differential Equation

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx}\right] + n(n+1)y = 0$$

$$y = c_1 P_n(x) + c_2 Q_n(x)$$

where P_n and Q_n are the Legendre functions and

$$P_0 = 1,\ P_1 = x,\ P_2 = \frac{1}{2}(3x^2 - 1),\ P_3 = \frac{1}{2}(5x^3 - 3x)$$

Associated Legendre's Differential Equation

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+[n(n+1)-\frac{m^2}{\left(1-x^2\right)}]y=0$$

$$y=c_1P_n^m(x)+c_2Q_n^m(x)$$

where P_n^m and Q_n^m are the associated Legendre functions where

$$P_1^1=(1-x^2)^{1/2}, \qquad P_2^1=3x(1-x^2)^{1/2}, \qquad P_2^2=3(1-x^2).$$

Also

$$P_n^0=P_n.$$

Hermite's Differential Equation

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2ny=0$$

The solutions are the Hermite polynomials, $H_n(x)$. These are

$$H_o(x)=1, \qquad H_1(x)=2x \qquad H_2(x)=4x^2-2, \qquad H_3(x)=8x^3-12x.$$

Laguerre's Differential Equation

$$\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+ny=0$$

The solutions are the Laguerre polynomials, $L_n(x)$. These are

$$L_o(x)=1, \qquad L_1(x)=1-x \qquad L_2(x)=x^2-4x+2.$$

Associated Laguerre's Differential Equation

$$x\frac{d^2y}{dx^2}+(m+1-x)\frac{dy}{dx}+(n-m)y=0.$$

Solutions are the associated Laguerre polynomials $L_n^m(x)$:

$$L_1^1=-1, \qquad L_2^1=2x-4, \qquad L_2^2=2, \qquad L_3^1=-3x^2+18x-18,$$
$$L_3^2=-6x+18, \qquad L_3^3=-6.$$

Chebyshev's Differential Equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0\,.$$

Solutions are the Chebyshev polynomials given as

$$T_n(x)=\cos\left(n\cos^{-1}x\right), \qquad T_0=1,\; T_1=x,\; T_2=2x^2-1,\; T_3=4x^2-3x$$

$$y=\begin{cases} c_1T_n(x)+B\sqrt{1-x^2}\,U_{n-1}(x) & n\neq 0 \\ A+B\sin^{-1}x & n=0 \end{cases}$$

Here $U_n(x)$ is the Chebyshev polynomials of the second kind.

Hypergeometric Differential Equations

$$x(1-x)\frac{d^2y}{dx^2}+\left[c-(a+b+1)x\right]\frac{dy}{dx}-aby=0$$

Solution is the hypergeometric function $F(a,b;c;x)$.

Fourier Series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx\,,\qquad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx$$

Fourier Cosine Series

When $f(x)=f(-x)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}\,,\qquad a_n=\frac{2}{L}\int_{0}^{L}f(x)\cos\frac{n\pi x}{L}dx$$

Fourier Sine Series

When $f(x)=-f(-x))$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}\,,\quad b_n=\frac{2}{L}\int_{0}^{L}f(x)\sin\frac{n\pi x}{L}dx$$

Fourier Exponential Series

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{i\omega_n x}\,,\qquad \text{where}\qquad \omega_n=\frac{n\pi}{L}\,,$$

$$c_n=\frac{1}{2L}\int_{-L}^{L}f(x)e^{-i\omega_n x}dx$$

Summary of Properties of Fourier Transforms

Consider the Fourier Exponential Series in the region $-L<x<L$

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{\frac{in\pi x}{L}}\qquad -L<x<L\,,$$

$$c_n=\frac{1}{2L}\int_{-L}^{L}e^{-\frac{in\pi x}{L}}f(x')dx$$

Replacing the expression for the coefficient c_n in the series, we find

$$f(x)=\frac{1}{2L}\sum_{-\infty}^{+\infty}\int_{-L}^{L}e^{i\omega_n(x-x')}f(x')dx\,.$$

In the last step we have defined

$$\omega_n=\frac{n\pi}{L},\qquad \Delta\omega=\frac{\pi}{L}\,.$$

As $L\to\infty$, $\sum g_n\Delta\omega=\int g d\omega$. Thus,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{i\omega(x-x')}f(x')dx'd\omega \qquad \text{Fourier Integral Representation}$$

Define Fourier Transform (Exponential)

$$\bar{f}(\omega)=\int_{-\infty}^{+\infty}e^{-i\omega x'}f(x')dx'$$

The inverse transform is

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\omega x}\bar{f}(\omega)d\omega$$

The above two equations are a Fourier Exponential Transform Pair.

Fourier Integral Representation (FIR) may be restated as

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cos\omega(x-x')f(x')dx'd\omega,$$

or

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\cos\omega x\cos\omega x'+\sin\omega x\sin\omega x')f(x')dx'd\omega\,.$$

For even functions (i.e. $f(x)=f(-x)$), and FIR becomes

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}\cos\omega x\cos\omega x' f(x')dx'd\omega$$

Definition: Fourier-Cos Transform Pair:

$$\bar{f}_c(\omega)=\int_{0}^{\infty}\cos\omega x' f(x')dx'$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos\omega x \overline{f_c}(\omega) d\omega$$

For odd functions (i.e. $f(x) = -f(-x)$), and FIR becomes

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} \sin\omega x \sin\omega x' f(x') dx'$$

Definition: Fourier-Sin Transform Pair:

$$\overline{f_s}(\omega) = \int_0^{\infty} \sin\omega x' f(x') dx'$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin\omega x \overline{f_s}(\omega) d\omega$$

Applications to Differential Equations

Fourier Exponential Transform of derivatives

$$\Im\left\{\frac{df}{dx}\right\} = \int_{-\infty}^{+\infty} e^{-i\omega x} \frac{df(x)}{dx} dx = i\omega\bar{f}(\omega)$$

$$\Im\left\{\frac{d^2 f}{dx^2}\right\} = -\omega^2 \bar{f}(\omega), \quad \Im\left\{\frac{d^n f}{dx^n}\right\} = (i\omega)^n \bar{f}(\omega)$$

Example: Find f that satisfies the following differential equation:

$$\frac{d^2 f}{dx^2} + a\frac{df}{dx} + bf = \delta(x - x_0) \qquad -\infty < x < +\infty$$

Take Fourier Exponential Transform

$$-\omega^2\bar{f}(\omega) + ai\omega\bar{f}(\omega) + b\bar{f}(\omega) = e^{-i\omega x_0}$$

$$\bar{f}(\omega) = \frac{e^{-i\omega x_0}}{b - \omega^2 + ia\omega}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\omega(x - x_0)}}{b - \omega^2 + ia\omega} d\omega$$

Laplace Transform

Consider the class of functions, which are zero for negative x, and are defined according to

$$f(x) = \begin{Bmatrix} F(x)e^{-\gamma x} & x \ge 0 & \gamma > 0 \\ 0 & x < 0 & \end{Bmatrix}$$

FIR of $f(x)$ for $x \ge 0$ becomes

$$F(x)e^{-\gamma x} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_0^{\infty} e^{i\omega(x-x')} F(x') e^{-\gamma x'} dx'$$

or

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega e^{(\gamma+i\omega)x} \int_0^{\infty} e^{-(\gamma+i\omega)x'} F(x') dx'$$

Let

$$\gamma + i\omega = S, \qquad d\omega = \frac{ds}{i}$$

$$F(x) = \frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} ds e^{sx} \int_0^{\infty} e^{-sx'} F(x') dx'$$

Definition: Laplace Transform

$$\bar{F}(s) = \int_0^{\infty} e^{-sx'} F(x') dx'$$

Inverse Transform

$$F(x) = \frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} e^{sx} \bar{F}(s) ds$$

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Table of Fourier Exponential Transform Pair

$$\bar{f}(\omega)=\int_{-\infty}^{+\infty} e^{-i\omega x} f(x)dx\,, \qquad f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega x}\bar{f}(\omega)d\omega$$

$f(x)$	$\bar{f}(\omega)$
$f_1(x)e^{-i\omega_0 x}$	$\bar{f}_1(\omega+\omega_0)$
$f_1(x+x_0)$	$e^{i\omega x_0}\bar{f}(\omega)$
$f_1(x)*f_2(x)=\int_{-\infty}^{+\infty} f_1(\xi)f_2(x-\xi)d\xi$	$\bar{f}_1(\omega)\bar{f}_2(\omega)$
$\delta(x-x_0)$	$e^{-i\omega x_0}$
$e^{i\omega_0 x}$	$2\pi\delta(\omega-\omega_0)$
$e^{-\alpha\|x\|}$	$\frac{2\alpha}{\omega^2+\alpha^2}$
$\cos\omega_0 x$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$e^{-\alpha\|x\|}\cos\beta x$	$\frac{2\alpha(\omega^2+\alpha^2+\beta^2)}{(\omega^2-\beta^2-\alpha^2)^2+4\alpha^2\omega^2}$
$e^{-\alpha\|x\|}\left[\cos\beta x+\frac{\alpha}{\beta}\sin\beta\|x\|\right]$	$\frac{4\alpha(\alpha^2+\beta^2)}{(\omega^2-\beta^2-\alpha^2)^2+4\alpha^2\omega^2}$
$e^{-\alpha^2x^2}\cos\beta x$	$\frac{\sqrt{\pi}}{2\alpha}\left[\exp\left\{-\frac{(\omega+\beta)^2}{4\alpha^2}\right\}+\exp\left\{-\frac{(\omega-\beta)^2}{4\alpha^2}\right\}\right]$
$e^{-\alpha^2x^2}$	$\frac{\sqrt{\pi}}{\alpha}\exp\left\{-\frac{\omega^2}{4\alpha^2}\right\}$
$\frac{d^n}{dx^n}\delta(x)$	$(i\omega)^n$
$J_0(x)$	$\begin{cases}\frac{2}{\sqrt{1-\omega^2}} & \|\omega\|<1\\ 0 & \text{elsewhere}\end{cases}$

Table of Laplace Transform Pair

$F(x)$	$\bar{F}(s)$
$e^{ax}F(x)$	$\bar{F}(s-a)$
$F(x-a)U(x-a)$	$e^{-as}\bar{F}(s)$
$\frac{dF}{dx}$	$s\bar{F}(s)-F(0)$
$\frac{d^nF}{dx^n}$	$s^n\bar{F}(s)-s^{n-1}F(0)\ldots-F_{(0)}^{(n-1)}$
$xF(x)$	$\frac{d\bar{F}(s)}{ds}$
$x^nF(x)$	$(-1)^n\frac{d^n\bar{F}(s)}{ds^n}$
$\int_0^x F(t)dt$	$\frac{\bar{F}(s)}{s}$
$F_1(x)*F_2(x)=\int_0^x F_1(x-t)F_2(t)dt$	$\bar{F}_1(s)\bar{F}_2(s)$
$\frac{F(x)}{x}$	$\int_s^\infty \bar{F}(s_1)ds_1$
$1,\ x,\ x^n$	$\frac{1}{s},\ \frac{1}{s^2},\ \frac{n!}{s^n}$
$e^{ax},\ \sin ax,\ \cos ax$	$\frac{1}{s-a},\ \frac{a}{s^2+a^2},\ \frac{s}{s^2+a^2}$
$\sinh ax,\ \cosh ax$	$-\frac{a}{s^2-a^2},\ \frac{s}{s^2-a^2}$
$x\sin ax,\ x\cos ax$	$\frac{2as}{(s^2+a^2)^2},\ \frac{s^2-a^2}{(s^2+a^2)^2}$
$J_0(ax),\ I_0(ax)$	$\frac{1}{\sqrt{s^2+a^2}},\ \frac{1}{\sqrt{s^2-a^2}}$
$\frac{\cos 2\sqrt{ax}}{\sqrt{\pi x}},\ \frac{\sin 2\sqrt{ax}}{\sqrt{\pi x}}$	$\frac{e^{-a/s}}{\sqrt{s}},\ \frac{e^{-a/s}}{s^{3/2}}$
$\delta(x-a),\ U(x-a)$	$e^{-as},\ \frac{e^{-as}}{s}$
$\frac{1}{\sqrt{\pi x}}e^{-\frac{a^2}{4x}}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$

Probability and Random Processes

In this section, capital letters identifies a random variable and lower case letters are used for coordinate systems.

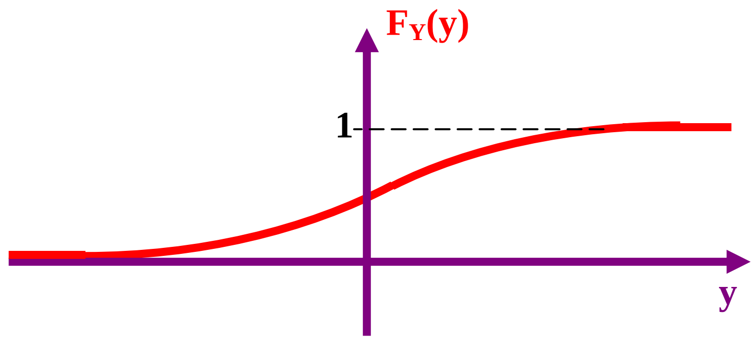
Distribution Function

The distribution function of a random variable Y is defined as the probability that $\{Y \le y\}$. That is

$$F_Y(y) = P\{Y \le y\}$$

It then follows that $F_Y(y)$ is monotonically increasing function and

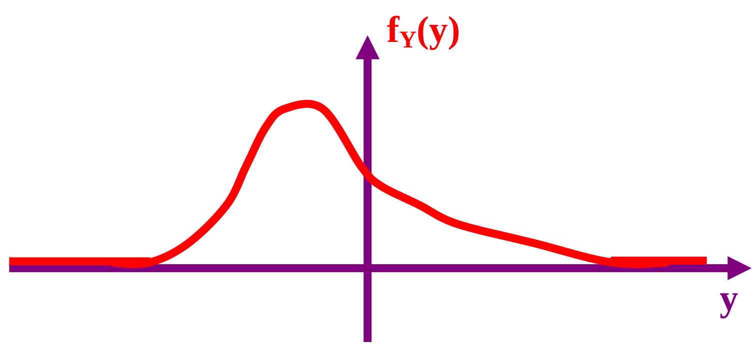
$$0 \le F_Y(y) \le 1.$$



Density Function

The probability density function is defined as

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$



Properties:

$$F_Y(\infty) = 1 = \int_{-\infty}^{+\infty} f_Y(y)dy\,, \qquad P\{y_1 < Y \le y_2\} = \int_{y_1}^{y_2} f_Y(y)dy = F_Y(y_2) - F_Y(y_1)$$

Expected Value

$$E\{Y\} = \overline{Y} = \int_{-\infty}^{+\infty} y f_Y(y)dy$$

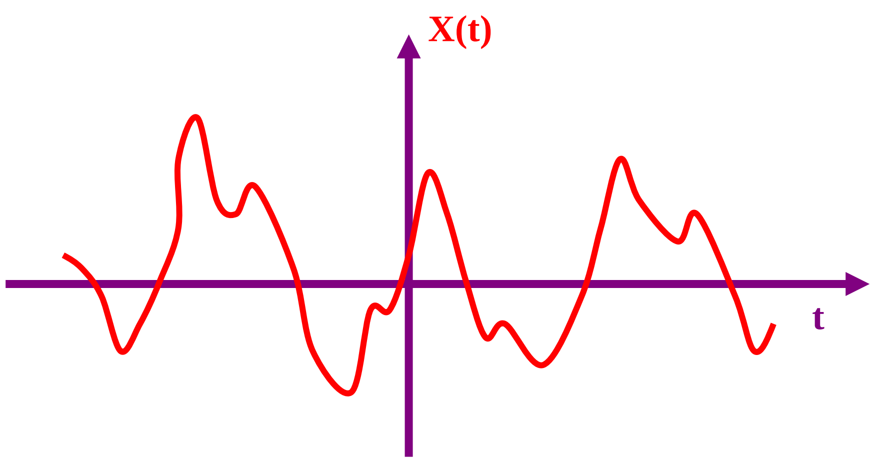
$$E\{g(Y)\} = \overline{g(Y)} = \int_{-\infty}^{+\infty} g(y) f_Y(y)dy$$

Variance

$$\sigma_Y^2 = E\left\{\left(Y - \overline{Y}\right)^2\right\} = E\left\{Y^2\right\} - \overline{Y}^2$$

Stochastic Process

Ensembles of random functions of time (or space) are referred to as stochastic processes. For fixed time, a stochastic process becomes a random variable. Every sample of a stochastic process is a time function.



Statistics of a stochastic process may be evaluated similar to those of a random variable. For example, the mean value is given as

$$E\{X(t)\} = \int_{-\infty}^{+\infty} x f_X(x,t)\,dx$$

Time Average

Time averaging over an interval (0,T) is defined as

$$\overline{X}(t) = \frac{1}{T}\int_0^T X(t)\,dt \approx E\{X(t)\}$$

Autocorrelation

The autocorrelation of a random process is defined as

$$R_{xx}(\tau) = E\{X(t+\tau)X(t)\} = \frac{1}{T}\int_0^T X(t+\tau)X(t)\,dt$$

where τ is the time difference, an it is assumed that X(t) is a stationary random process. Note that

$$R_{xx}(0) = E\{X^2(t)\} = \overline{X^2(t)}$$

Energy Spectrum

$$S_{xx}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\tau} R_{xx}(\tau) d\tau$$

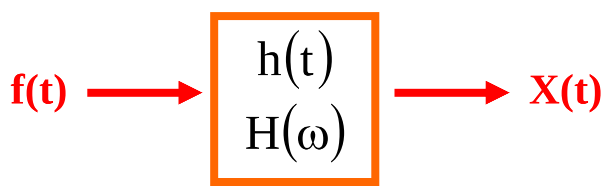
$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega\tau} S_{xx}(\omega) d\omega$$

It may be shown that

$$S_{xx}(\omega) = \frac{1}{T} \left| \tilde{X}(\omega) \right|^2, \qquad \tilde{X}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} X(t) dt$$

Linear Systems

Consider a linear system with impulse response h(t) and a system function $H(\omega)$ for $X(0) = 0$ as shown schematically in the figure.



The solution then is given as

$$X(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

where

More generally,

$$X(t) = \int_{-\infty}^{+\infty} h(t-\tau) f(\tau) d\tau = h(t) * f(t)$$

Taking Fourier Transform $\qquad H(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} h(t) dt$

$$\tilde{x}(\omega) = H(\omega)\tilde{f}(\omega)$$

$$S_{xx}(\omega) = \frac{1}{T}\left|\bar{x}(\omega)\right|^2 = \frac{1}{T}\left|H(\omega)\right|^2\left|\bar{f}(\omega)\right|^2 = \left|H(\omega)\right|^2 S_{ff}(\omega)$$

$$S_{xx}(\omega) = \left|H(\omega)\right|^2 S_{ff}(\omega)$$

For

$$\dot{x} + \alpha x = f(t),$$

$$h(t) = e^{-\alpha t}$$

For

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = f(t),$$

$$h(t) = \frac{1}{\omega_d} e^{-\zeta\omega_0 t} \sin\omega_d t \qquad \omega_d = \omega_0\sqrt{1-\zeta^2}$$

Useful Integrals

$$\int e^{ax}P(x)dx = \frac{e^{ax}}{a}\left[P(x) - \frac{1}{a}P'(x) + \frac{1}{a^2}P''(x) - \frac{1}{a^3}P'''(x) \ldots\right]$$

$$\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

$$\int e^{ax}\cos bx dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$\int x\sin ax dx = \frac{\sin ax - ax\cos ax}{a^2}$$

$$\int x\cos ax dx = \frac{\cos ax + ax\sin ax}{a^2}$$

$$\int \ln ax dx = x\ln ax - x$$

$$\int x\ln ax dx = \frac{x^2}{2}\ln ax - \frac{x^2}{4}$$

$$\int x\sinh ax dx = \frac{ax\cosh ax - \sinh ax}{a^2}$$

$$\int x\cosh ax dx = \frac{ax\sinh ax - \cosh ax}{a^2}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \qquad \int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) = \frac{1}{2a}\ln\frac{x-a}{x+a}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int e^{ax}x dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$

Vector Identities

$$\nabla \cdot \nabla \times \vec{u} = 0 \qquad \nabla \times (\nabla \varphi) = 0$$

$$\nabla \times \nabla \times \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$$

$$\vec{u} \cdot \nabla \vec{u} = \nabla\left(\frac{u^2}{2}\right) - \vec{u} \times (\nabla \times \vec{u})$$

$$\nabla \times (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \vec{u} - \vec{u} \cdot \nabla \vec{v} + (\nabla \cdot \vec{v})\vec{u} - (\nabla \cdot \vec{u})\vec{v}$$

$$\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \times \vec{u} - \vec{u} \cdot \nabla \times \vec{v}$$

$$\nabla(\vec{u} \cdot \vec{v}) = \vec{v} \cdot \nabla \vec{u} + \vec{u} \cdot \nabla \vec{v} + \vec{v} \times (\nabla \times \vec{u}) + \vec{u} \times (\nabla \times \vec{v})$$

Stokes Theorem

$$\oint_c \vec{u} \cdot d\vec{c} = \int_s (\nabla \times \vec{u}) \cdot ds$$

Divergence Theorem

$$\int_V \nabla \cdot \vec{u} dV = \int_S \vec{u} \cdot d\vec{s}$$