Brownian Motion

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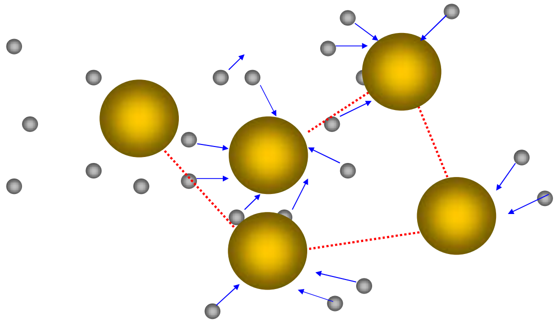
Outline

- **Introduction to Aerosols**
- **Drag Forces**
- **Cunningham Corrections**
- **Lift Forces**
- **Brownian Motion**
- Particle Deposition Mechanisms
- Gravitational Sedimentation
- Aerosol Coagulation

Brownian Motion



A Particle under Random Molecular Impact

Brownian Motion

Langevin Equation →

$$\frac{du}{dt}+\beta u = n(t)$$

$$\beta = 3\pi\mu d / C_c m = 1/\tau$$

N(t) = White Noise

Spectral Intensity →

$$S_{nn} = \frac{2kT\beta}{\pi m}$$

Brownian Motion - Solving a Stochastic Equation

Autocorrelation Function

$$R(\tau) = \overline{u(t+\tau)u(t)}$$

Fourier Transform

$$R_{uu}(\tau) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{i\omega\tau} S_{uu}(\omega)d\omega$$

$$S_{uu}(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-i\omega\tau} R_{uu}(\tau)d\tau$$

Brownian Motion - Solving a Stochastic Equation

$$S_{uu}(\omega) = |H(\omega)|^2 S_{nn}(\omega)$$

System Function → $H(\omega) = \frac{1}{i\omega+\beta}$

Response Power Spectrum → $S_{uu}(\omega) = \frac{2kT\beta/\pi m}{\omega^2+\beta^2}$

Autocorrelation → $R_{uu}(\tau) = \frac{kT}{m}e^{-\beta|\tau|}$

Brownian Motion

Mass Diffusivity → $D = \frac{1}{2}\frac{d}{dt}\overline{x^2}(t)$

Position → $x(t) = \int_0^t u(t_1)dt_1$

Variance → $\overline{x^2(t)} = \int_0^t\int_0^t R_{uu}(\tau_1-\tau_2)d\tau_1 d\tau_2$

Brownian Motion

Variance → $\overline{x^2(t)} = 2\int_0^t (t-\tau)R_{uu}(\tau)d\tau$

Diffusivity → $D = \int_0^\infty R_{uu}(\tau)d\tau$

→ $D = \frac{kT}{\beta m} = \frac{kTC_c}{3\pi\mu d}$

Brownian Motion -Markov Process
Fokker-Plank Approach

Fokker-Planck Equation

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial u}(\beta u f) = \frac{kT\beta}{m}\frac{\partial^2 f}{\partial u^2}$$

Probability Density

$$f = \frac{1}{\sqrt{2\pi kT/m}}\, e^{-\frac{mu^2}{2kT}}$$

Brownian Motion in a Force Field

Langevin Equation

$$\ddot{x} + \beta\dot{x} - \frac{F(x)}{m} = n(t)$$

$$F(x) = -\frac{\partial V(x)}{\partial x}$$

Fokker-Planck Equation

$$\frac{\partial f}{\partial t} = -\frac{\partial(\dot{x}f)}{\partial x} + \frac{\partial}{\partial \dot{x}}[(\beta\dot{x} - \frac{1}{m}F(x))f] + \frac{kT\beta}{m}\frac{\partial^2 f}{\partial \dot{x}^2}$$

Brownian Motion in a Force Field

Probability Density

$$f = C_0 \exp\{-\frac{m}{kT}[\frac{\dot{x}^2}{2} - \int_0^x \frac{F(x_1)dx_1}{m}]\}$$

$$f = C_0 \exp\{-\frac{1}{kT}[\frac{m\dot{x}^2}{2} + V(x)]\}$$

Gravitational Field

$$f = C_0 e^{-\frac{m\dot{x}^2}{2kT}}\, e^{-\frac{mg(x-x_0)}{kT}}$$

Computer Simulation Procedure

White Noise

$$\overline{n(t)} = 0$$

$$\overline{n(t_1)n(t_2)} = 2\pi S_{nn}\delta(t_1 - t_2)$$

- **Choose a time step ($\Delta t \ll \tau$)**
- **Generate a sequence of uniform random numbers ($0 < U < 1$)**
- **Transform to Gaussian random numbers.**

Computer Simulation Procedure

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- $$G_1 = \sqrt{-2\ln U_1}\cos 2\pi U_2$$
  $$G_2 = \sqrt{-2\ln U_1}\sin 2\pi U_2$$
- **Amplitude of the Brownian force is given by** $$n(t_i) = G_i\sqrt{\frac{\pi S_{nn}}{\Delta t}}$$
- **The generated sample of Brownian force need to be shifted by** $U\Delta t$

Sample Simulated White Noise Process

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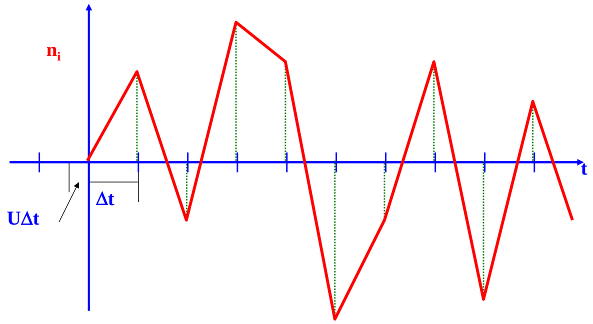

Sample Simulated White Noise Process

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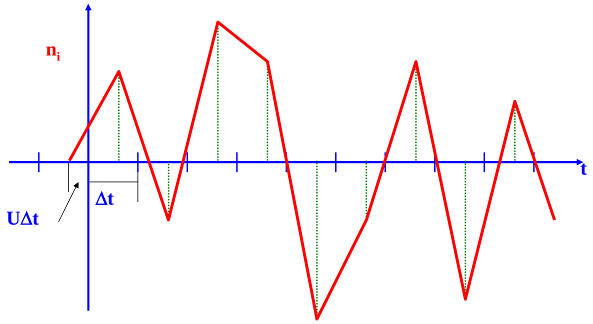

Particle Dispersion and Deposition in Viscous Sublayer

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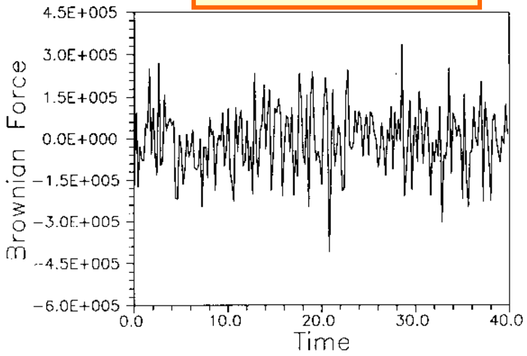
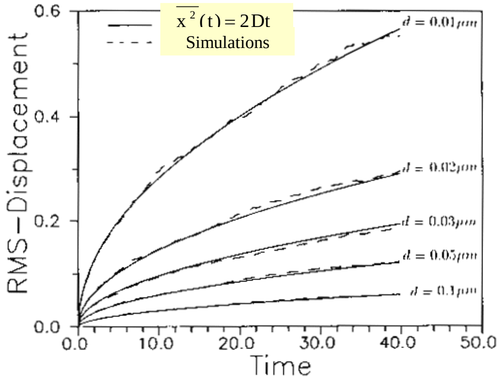
Brownian Force

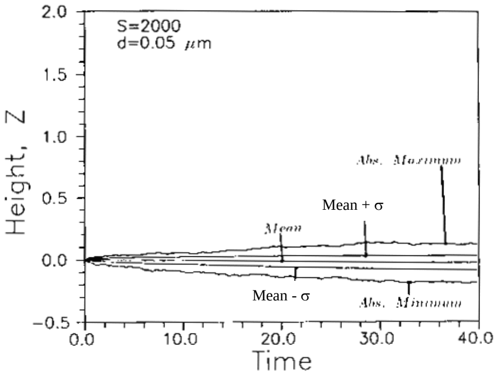


Ounis, Ahmadi and McLaughlin (1991)

Particle Dispersion in Viscous Sublayer



Particle Dispersion in Viscous Sublayer



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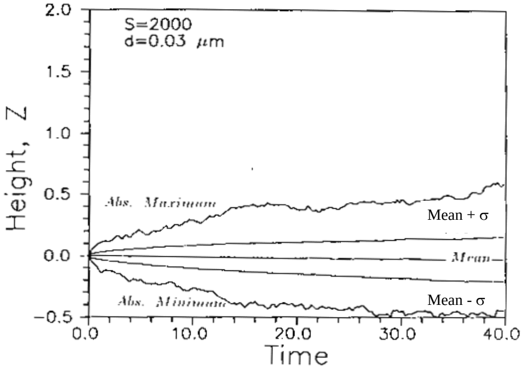

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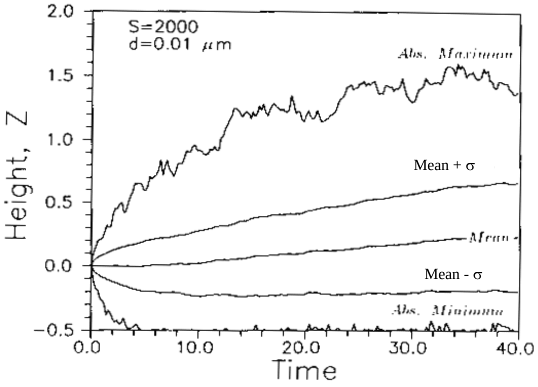
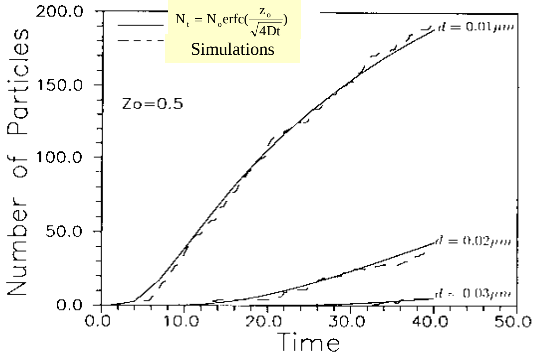


Particle Dispersion in Viscous Sublayer



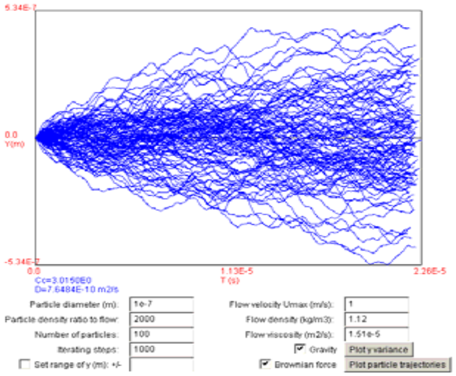
Java Applet for Brownian Motion

Equation of Motion

$$\frac{d\mathbf{u}^p}{dt} = \frac{1}{\tau}(\mathbf{u}^f - \mathbf{u}^p) + \mathbf{g} + \mathbf{n}(t)$$

Variance → $\sigma_y^2(t) = 2Dt$

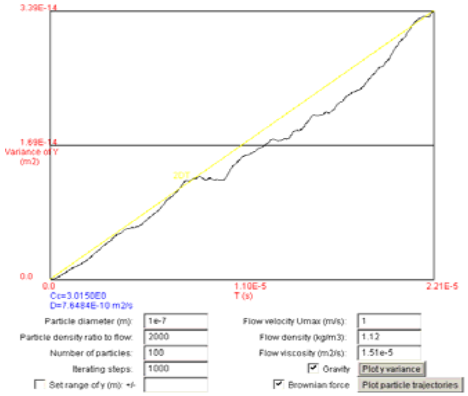
Particle Dispersion -Java Applet for particle trajectory analysis



Cc=3.0150E0
D=7.6484E-10 m2/s

Particle diameter (m): 1e-7
Particle density ratio to flow: 2000
Number of particles: 100
Iterating steps: 1000
☐ Set range of y (m): +/-
Flow velocity Umax (m/s): 1
Flow density (kg/m3): 1.12
Flow viscosity (m2/s): 1.51e-5
☑ Gravity
☑ Brownian force

Particle Dispersion -Java Applet for particle trajectory analysis



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