Review for Final Exam

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Outline

- Hydrodynamic Forces and Moments
- Diffusion Mechanisms
- Particle Adhesion and Detachment
- Particle Charging
Hydrodynamic Forces

Drag Forces

\[ F_D = \frac{3\pi\mu Ud_f}{C_C C_D} \]

Cunningham Correction

\[ C_D = \frac{24[1 + 0.15 \text{Re}^{0.687}]}{\text{Re}} \]

\[ C_C = 1 + \frac{2\lambda}{d} \left[ 1.257 + 0.4e^{-1.1d/2\lambda} \right] \]
Lift Force

\[ F_{L(Saff)} = 1.615 \rho v^{1/2} d^2 (u^f - u^p) \left| \frac{du^f}{dy} \right|^{1/2} \text{sgn}(\frac{du^f}{dy}) \]

Saffman (1965, 1968)

\[ \text{Sgn}(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases} \]
Equation of Motion

\[ m \frac{du^p}{dt} = \frac{3\pi \mu d}{C_c} (u^f - u^p) + mg \]
\[
\tau \frac{du^p}{dt} = (u^f - u^p) + \tau g
\]

**Relaxation Time**

\[
\tau = \frac{mC_c}{3\pi\mu d} = \frac{d^2\rho^p C_c}{18\mu} = \frac{Sd^2C_c}{18\nu}
\]

\[
S = \frac{\rho^p}{\rho^f}
\]

\[
\tau(s) \approx 3 \times 10^{-6} d^2 (\mu m)
\]
Turbulent stress is negligible

\[ \tau_0 = \mu \frac{dU}{dy} \]

\[ \frac{dU^+}{dy^+} = 1 \]

\[ u^+ = y^+ \]

\[ u = \frac{u^+ y}{y^+} \]

\[ 0 < y^+ \leq 5 \]

\[ F^+_{L(Saff)} = 0.807 d^+^3 \]

\[ U^k = \frac{1}{2} U_0 \]

\[ U^k \sim \text{Shear velocity} \]

\[ \gamma, u^* \]

\[ F^+ = \frac{F}{\mu y} \]
Fick’s Law: $J = -D \frac{dc}{dx}$

Diffusion Equation: \( \frac{\partial c}{\partial t} + \mathbf{v} \cdot \nabla c = D \nabla^2 c \)

Diffusivity: $D = \frac{kT C_c}{3\pi\mu d}$
Diffusion

- Similarity Method
- Separation of Variable Method
- Integral Method
Particle Adhesion and Detachment

- van der Waals Force
- JKR Adhesion Model
- DMT Adhesion Model
- Maugis-Pollock Model
- Particle Detachment Mechanisms
- Maximum Moment Resistance
\[ a^3 = \frac{d}{2K} \left[ P + \frac{3}{2} W_A \pi d + \sqrt{3\pi W_A dP + \left(\frac{3\pi W_A d}{2}\right)^2} \right] \]

\[ K = \frac{4}{3} \left[ \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right]^{-1} \]

\[ a^3 = \frac{dP}{2K} \]
Derjaguin-Muller-Toporov (1975)

**Pull-Off Force**

\[ F_{Po}^{DMT} = \pi W_A d \]

**Contact Radius at Zero Force**

\[ F_{Po}^{DMT} = \frac{4}{3} F_{Po}^{JKR} \]

\[ a_0 = \left( \frac{\pi W_A d^2}{2K} \right)^{\frac{1}{3}} \]

**Contact Radius at Separation**

\[ a = 0 \]
\[ P + \pi W_A d = \pi a^2 H \]

Elastic:
\[ a_0 \sim d^{\frac{2}{3}} \]

Plastic:
\[ a_0 \sim d^{\frac{1}{2}} \]
\[ a^* \left(1 - P^* + \sqrt{1 - 2P^*} \right)^{1/3} \]

\[ P^* = -\frac{P}{\frac{3}{2} \pi W_A d} \]

\[ a^* = \frac{a}{\left(\frac{3\pi W_A d^2}{4K}\right)^{1/3}} \]

\[ M_{\text{JKR}}^* = P^* a^* = P^* \left(1 - P^* + \sqrt{1 - 2P^*} \right)^{1/3} \]

\[ M_{\text{JKR}}^{\text{max}} = 0.42 \]
Coulomb Force

\[ F_E = qE \]

\[ q = ne \]

Particle Mobility

\[ u = Z^p = \frac{qC_c}{3\pi\mu d} \]
Boltzmann Equilibrium
Charge Distribution

$$f(n) = \frac{0.24}{\sqrt{d\pi}} \exp\left\{-\frac{0.05n^2}{d}\right\}$$

$$\bar{n} \approx 2.36\sqrt{d}, \quad d(\mu m)$$

$d > 0.02\mu m$
Diffusion Charging

\[ n = \frac{dkT}{2e^2} \ln\left[1 + \left(\frac{2\pi}{m_i kT}\right)^{1/2} n_{i\infty}de^2 t\right] \]

Field Charging

\[ n_{\infty} = \left[1 + \frac{2(\varepsilon_p - 1)}{\varepsilon_p + 2}\right] \frac{Ed^2}{4e} \text{ as } t \to \infty \]
(40 points) Consider a steady convective-diffusion process with a flow velocity near an absorbing wall. The governing equation is given by

\[ ay^2 \frac{\partial C}{\partial x} = D \frac{\partial^2 C}{\partial y^2} \]

where D is the diffusivity and a is a constant. The boundary conditions are:

\[ C(x,0) = 0 \quad C(0,y) = C_0, \quad \text{and} \quad C(x,\infty) = C_0 \]

i. Use a similarity variable \( \eta = \frac{y}{2(Dx/a)^{1/4}} \) reduce the governing equation and boundary conditions to the similarity form.

ii. Evaluate the concentration profile and the deposition velocity to the wall.
\[ a \gamma^2 \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial x^2} \quad c = c(\eta) \]

\[ \frac{x}{\eta^2} \sim \frac{y^2}{\gamma^2} \quad \left( \frac{y^2}{\gamma^2} \right) \sim \eta \]

\[ \frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial c}{\partial \eta} \left( \frac{1}{2(DX/a)^{1/2}} \right) \]

\[ \frac{\partial^2 c}{\partial \eta^2} = \frac{\partial^2 c}{\partial x \partial \eta} \frac{1}{4(DX/a)^{1/2}} \]

\[ \frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial c}{\partial \eta} \left( -\frac{1}{4} \frac{\eta}{x} \right) \]

\[ a \gamma^2 (-\frac{1}{4} \frac{\eta}{x}) \frac{\partial c}{\partial \eta} = D \frac{\partial^2 c}{\partial \eta^2} \frac{1}{4(DX/a)^{1/2}} \]

\[ -\frac{a y^2}{x D} \left( \frac{DX}{a} \right)^{1/2} \frac{\partial c}{\partial \eta} = \frac{\partial^2 c}{\partial \eta^2} \rightarrow -4 \eta^3 \frac{\partial c}{\partial \eta} = \frac{\partial^2 c}{\partial \eta^2} \]
\[
\frac{d^2 c}{d \eta^2} + 4 \eta^3 \frac{dc}{d \eta} = 0 \quad C(0) = 0, \quad C(\infty) = C_0
\]

\[
\frac{d^2 c}{d \eta^2} = -4 \eta^3 \rightarrow \frac{dc}{d \eta} = -\eta^4 + \lambda K
\]

\[
\frac{dc}{d \eta} = K e^{-\eta^4} \quad C(0) = 0
\]

\[
c = K \int_{\eta}^{\infty} e^{-\eta^4} d\eta + A, \quad C_0 = K \int_{0}^{\infty} e^{-\eta^4} d\eta
\]

\[
c = C_0 \frac{\int_{0}^{\eta} e^{-\eta^4} d\eta}{\int_{0}^{\infty} e^{-\eta^4} d\eta}
\]

\[
J = D \frac{dc}{d \eta} \bigg|_{\eta = 0} = D \frac{dc}{d \eta} \bigg|_{\eta = \infty} = D \frac{C_0}{\int_{0}^{\infty} e^{-\eta^4} d\eta} \frac{1}{2(DX/\lambda)^{1/2}}
\]
\[ U_d = \frac{J}{C_0} = \frac{D}{2\left(\frac{AX}{a}\right)^{1/4}} \frac{1}{\int_0^a e^{-y/4} dy} \]
Problem 2

(35 points) Consider a 12 µm silicon particle that is attached to a silicon wafer in a turbulent air flow with a shear velocity of 2 m/s.

i. Evaluate the drag, the Saffman lift and the hydrodynamic moment acting on the particle in wall units and in SI units.

ii. Evaluate the pull-off force as predicted by the JKR model.

iii. Find the contact radius at zero force and at the separation according to the JKR model.

iv. Is the particle going to be removed by the rolling mechanism? (Assume $u^+ = y^+$, and for silicon use $W_A = 0.0389$ J/m², $E = 1.79 \times 10^{11}$ N/m², and Poisson ratio of 0.27. The kinematic viscosity of air is $\nu = 1.5 \times 10^{-5}$ m² / s)
\( d = 12 \text{ mm}, \quad U^y = 2 \text{ m/s}, \quad d^2 = \frac{D u^k}{\sqrt{g}} = \frac{12 \times 10^{-2} \times 2}{1.5 \times 10^{-5}} = 1.6 \)

\[
F_t = \frac{3 \pi \mu d U^y}{C_c} \quad (\text{Eq. 22})
\]

\[
C_D = 1 \quad R_e = \text{small}
\]

\[
F_t = \frac{3 \pi (1.7) \mu d U^y}{2 \gamma C_c}
\]

\[
= \frac{2.9 \pi (1.7) \mu d^2 U^y}{C_c} = \frac{2.9 \pi \mu d^2 U^y}{C_c} \quad (2a)
\]

\[
C_c = 1 \quad \text{for} \quad d = 12 \text{ mm}
\]

\[
F_t^+ = \frac{F_t}{\mu^2} = \frac{2.9 \pi d^2}{C_c} = 9.11 \quad d^2 = 23.22
\]

\[
F_t = \mu \nu \quad F_t^+ = 1.2 (1.5 \times 10^{-5}) 23.22 = 6.29 \times 10^{-9} \text{ N}
\]

\[
\mu = \frac{p \nu}{1.2}
\]
\[
\begin{align*}
F_L^+ &= 0.80 \times d^{-3} = 3.81, \quad F_L = M \frac{d}{F_L^+} = 8.92 \times 10^{-10} N \\
M &= 1.07 \pi R u^2 \frac{d^3}{c_c} \quad (\text{Eq. 30}) = 3.36 \rho u^2 d^3 \\
M^+ &= \frac{M}{\rho u^2 d} = 3.36 d^{+3} = 13.77 \\
M &= 2.79 \times 10^{-14} N \cdot m \\
\text{(i)} \quad F_e^n = \frac{3 \pi \epsilon_0 W A d}{4} = \frac{3}{4} \pi (0.0389) 1.2 \times 10^{-5} = 1.1 \times 10^{-6} N \\
\text{(ii)} \quad K = \frac{4}{3} \frac{E}{2(1-v^2)} = 1.287 \times 10^{11} N/m^2 \\
A_0 &= \left( \frac{3 \pi \epsilon_0 W A d^2}{2K} \right)^{1/3} = 5.8 \times 10^{-8} m = 0.059 \mu m \\
\alpha &= a_0 \sqrt[4]{15} = 0.0372 \mu m
\end{align*}
\]
14) \[ F_D \frac{1}{2} + M_x + F_L a \geq M_{JKR}^{\text{Max}} \]

\[ 3.32 \times 10^{-12} \]

\[ 3.78 \times 10^{-14} \]

\[ 2.79 \times 10^{-14} \]

\[ 4.1 \times 10^{-14} \]

\[ M_{\text{Hydro}} = 6.57 \times 10^{-14} \geq 4.1 \times 10^{-14} = M_{\text{Max}}^{\text{JKR}} \]

\[ \therefore \text{Particle is Removed!} \]
Problem 3

(25 points) Consider a cloud of 12 µm quartz particles with a concentration of $10^5$ particles per cm$^3$.

i. Find the average absolute number of charge for the equilibrium Boltzmann distribution.

ii. Determine the number of particles that will carry 5 positive charges. How many will carry no charges in this case?

iii. Find the mean electrostatic precipitation velocity for a field of 400 Volt/cm for particles with the average absolute charge distribution.

iv. Find the terminal velocity of these particles and compare with the electrostatic precipitation velocity.

(The density of air is 1.2 kg/m$^3$, the density ratio of quartz particle to air is 2000, and charge of electron is $1.59 \times 10^{-19}$ Coul.)
d = 12 \mu m, \ c_0 = 10^5 \#/cm^3

\text{i) } \bar{n} = 2.36 \sqrt{d} = 2.36 \sqrt{12} = 8.17

\text{ii) } f(n) = \frac{0.24}{\sqrt{\pi d}} e^{-0.058 n^2/d} \approx 0.0391 e^{-0.00483 n^2}

f(0) = 0.0391 \quad N_0 = 3910

f(5) = 0.0346 \quad N_0 = 3460

\text{iii) } \bar{n} = \frac{E_0 C_0}{3 \pi \mu d}
\[ U = \frac{40000 \times (8.17) \times (1.59 \times 10^{-19})}{3\pi \times (1.8 \times 10^{-5}) \times 1.2 \times 10^{-5}} = 2.55 \times 10^{-5} \text{ m/s} \]

\[ U = 25.5 \text{ mm/s} \]

iv) \[ U_t = 2g = \frac{Sd^2 g}{18} = \frac{2000 \times (12 \times 10^{-6})^2 \times 9.81}{18 \times (1.5 \times 10^{-5})} \]

\[ U_t = 1.05 \times 10^{-2} \text{ m/s} = 1.05 \text{ cm/s} \]