Incompressible Viscous Flows

For an incompressible fluid, the continuity equation and the Navier-Stokes equation are given as

$$\nabla \cdot \mathbf{v} = 0 , \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{v} . \tag{2}$$

Using a vector identity, Equation (2) may be restated as

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \frac{|v|^2}{2} - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nu \nabla \times \nabla \times \nabla \times \mathbf{v} \tag{3}$$

Define vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \tag{4}$$

and taking curl of (3) we find

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = -\nu \nabla \times \nabla \times \boldsymbol{\omega} . \tag{5}$$

Noting that curl of gradient is zero and

$$\nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = (\nabla \cdot \boldsymbol{\omega})\mathbf{v} - (\nabla \cdot \mathbf{v})\boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \boldsymbol{\omega} \tag{6}$$

Equation (5) may be restated as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{v} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{v} + \nu \nabla^2 \boldsymbol{\omega} \tag{7}$$

Equation (7) is the vorticity transport equation. It shows that in addition to being convected and diffused, vorticity is also generated by the first on the right hand side of Equation (7) by a vortex stretching mechanism.

Two-Dimensional Plane Flows

For two dimensional flows in xy-plane as shown in Figure 1, let

$$\mathbf{v} = \nabla \times (\mathbf{k}\psi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix} = \frac{\partial \psi}{\partial y}\mathbf{i} - \frac{\partial \psi}{\partial x}\mathbf{j}, \tag{8}$$

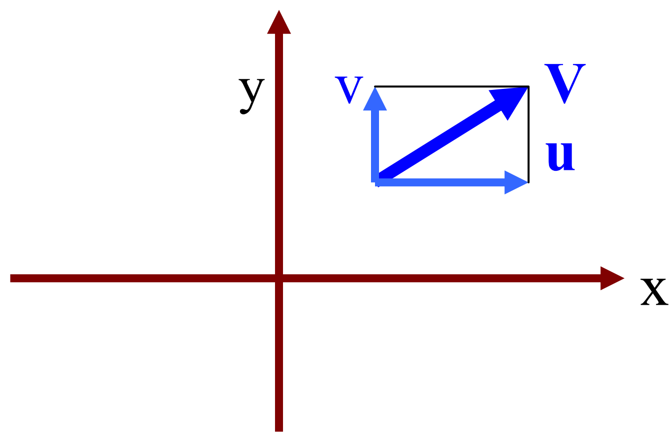

Figure 1. Schematics of plane flows in a Cartesian coordinate system.

That is

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{9}$$

and Equation (1) is satisfied.

The nonzero element of $\boldsymbol{\omega}$ is

$$\omega_z = \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2 \psi. \tag{10}$$

Equation (7) in two-dimensional case reduces to

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \nabla^4 \omega. \tag{11}$$

Using (10), Equation (11) may be restated as

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi = \nu\nabla^4\psi. \tag{12}$$

or

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} = \nu\nabla^4\psi . \tag{13}$$

Equation (13) is the equivalent to the Navier-Stokes equation and contains a single unknown ψ.

Plane Flows in a Cylindrical Geometry

Case (a) $v_z = 0$ and v_r and v_θ are functions of r and θ.

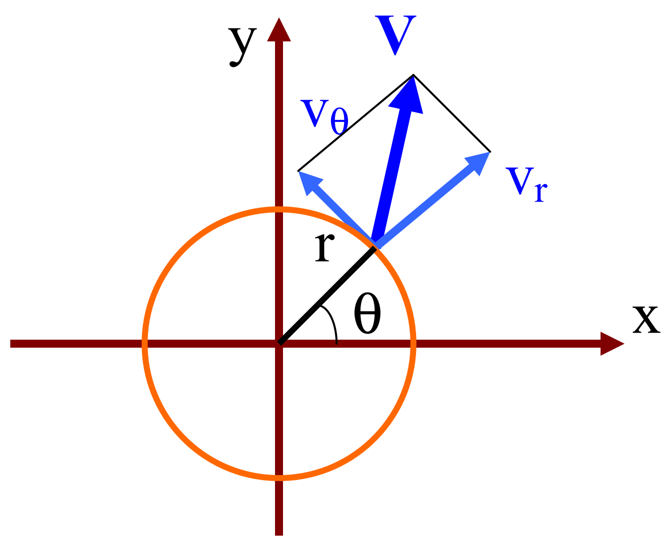

Figure 2. Schematics of plane flows in a polar coordinate system.

For a plane flow in cylindrical geometry as shown in Figure 2, let

$$\mathbf{v} = \nabla \times (\mathbf{e}_z \psi(r, \theta)). \tag{14}$$

That is,

$$\mathbf{v} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_r - \frac{\partial\psi}{\partial r}\mathbf{e}_\theta , \tag{15}$$

or

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = -\frac{\partial\psi}{\partial r} \tag{16}$$

The nonzero element of $\boldsymbol{\omega}$ is given by

$$\omega_z = \omega = -\nabla^2 \psi \tag{17}$$

where

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \tag{18}$$

Equation (7) now becomes

$$\frac{\partial \omega}{\partial t} + v_r \frac{\partial \omega}{\partial r} + v_\theta \frac{1}{r}\frac{\partial \omega}{\partial \theta} = \nu \nabla^2 \omega . \tag{19}$$

Using (17), Equation (19) may be restated as

$$\frac{\partial}{\partial t}\nabla^2 \psi + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial r}\nabla^2 \psi - \frac{\partial \psi}{\partial r}\frac{1}{r}\frac{\partial}{\partial \theta}\nabla^2 \psi = \nu \nabla^4 \psi , \tag{20}$$

or

$$\frac{\partial}{\partial t}\nabla^2 \psi + \frac{1}{r}\frac{\partial(\nabla^2 \psi, \psi)}{\partial(r, \theta)} = \nu \nabla^4 \psi . \tag{21}$$

Equation (21) is the equation governing $\psi(r, \theta)$ in plane flows expressed in polar coordinated system.

Case (b) $v_\theta = 0$, v_r and v_z are functions of r and z.

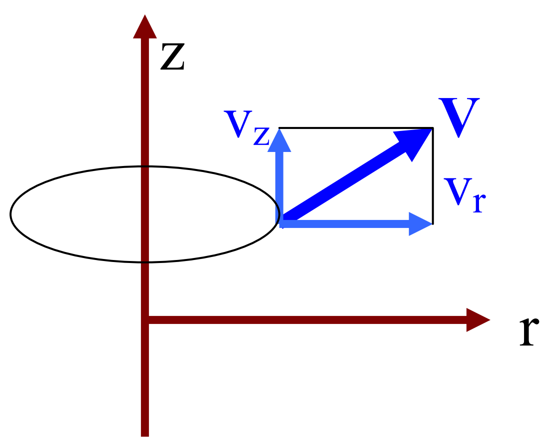

Figure 3. Schematics of axisymmetric flows in a cylindrical coordinate system.

For an axisymmetric flow in cylindrical coordinates, let

$$\mathbf{v} = \nabla \times \left(\mathbf{e}_\theta \frac{\psi}{r}(r, z) \right) \tag{22}$$

That is

$$\mathbf{v} = -\frac{1}{r}\frac{\partial \psi}{\partial z}\mathbf{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial r}\mathbf{e}_z, \tag{23}$$

or

$$v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}. \tag{24}$$

The vorticity define by Equation (4) now becomes

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = \mathbf{e}_\theta \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) = -\mathbf{e}_\theta \frac{1}{r}\left(\frac{\partial^2 \psi}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r} \right) \right) \tag{25}$$

Thus, the only nonzero component of vorticity is given by

$$\omega_\theta = \omega = -\frac{1}{r}E^2\psi, \tag{26}$$

where

$$E^2\psi = \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}. \tag{27}$$

Equation (7) may now be restated as

$$\frac{\partial \omega_\theta}{\partial t} + v_r \frac{\partial \omega_\theta}{\partial r} + v_z \frac{\partial \omega_\theta}{\partial z} - \frac{v_r \omega_\theta}{r} = -\nu \nabla \times \nabla \times (\omega_\theta \mathbf{e}_\theta)|_{\theta-\text{comp.}} \tag{28}$$

Using (26) in (28) we find

$$\frac{\partial}{\partial t}(E^2\psi) - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial}{\partial r}(E^2\psi) + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial}{\partial z}(E^2\psi) + \frac{2}{r^2}\frac{\partial \psi}{\partial z}E^2\psi = \nu E^4\psi \tag{29}$$

or

$$\frac{\partial}{\partial t}\left(E^2\psi\right)-\frac{1}{r}\frac{\partial\left(E^2\psi,\psi\right)}{\partial(r,z)}+\frac{2}{r^2}\frac{\partial\psi}{\partial z}E^2\psi=\nu E^4\psi \tag{30}$$

Equation (30) governs $\psi(r,z)$ in axisymmetric cylindrical flows.

Spherical Coordinates

Spherical coordinate system is shown in Figure 4. Here

$$\begin{cases} x = r\cos\theta\cos\varphi \\ y = r\cos\theta\sin\varphi \\ z = r\sin\theta \end{cases} \tag{31}$$

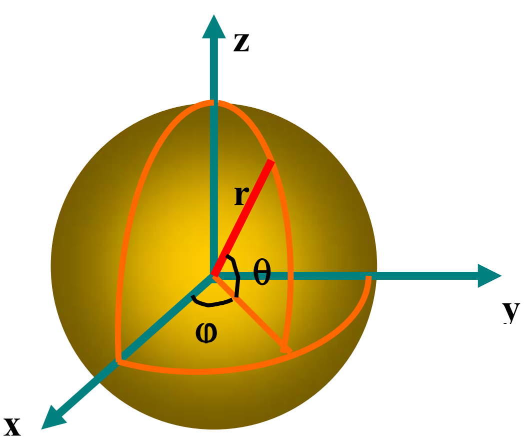

Figure 4. Schematics of spherical coordinate system.

Consider the case when $v_\varphi = 0$ and v_r and v_θ are only functions of r and θ.

Let

$$\mathbf{v} = \nabla\times\left(\frac{\mathbf{e}_\varphi\psi(r,\theta)}{r\sin\theta}\right), \tag{32}$$

That is

$$\mathbf{v}=\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\mathbf{e_r}-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\mathbf{e_\theta} \tag{33}$$

or

$$v_r=\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\ v_\theta=-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{34}$$

The vorticity equation will reduce to

$$\frac{\partial}{\partial t}\left(E^2\psi\right)+\frac{1}{r^2\sin\theta}\frac{\partial\left(E^2\psi,\psi\right)}{\partial(r,\theta)}+\frac{2E^2\psi}{r^2\sin^2\theta}\left(\frac{\partial\psi}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right)=\nu E^4\psi, \tag{35}$$

where

$$E^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}. \tag{36}$$

Intrinsic Coordinates

It is sometimes simpler to work with a coordinate system, which is attached to the surface of revolution. For the body of revolution shown Figure 5, consider the unit vectors $\mathbf{n}$, $\mathbf{s}$, $\mathbf{e}_\varphi$. For this system the metrics are

$$h_1=h_n=1,\qquad h_2=h_\varphi=\rho,\qquad h_3=h_s=1 \tag{37}$$

Thus

$$\nabla\Phi=\mathbf{n}\frac{\partial\Phi}{\partial n}+\mathbf{s}\frac{\partial\Phi}{\partial s}+\mathbf{e}_\varphi\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi} \tag{38}$$

The stream function for axisymmetric flows may now be introduced as

$$\mathbf{v}=\nabla\times\left(\mathbf{e}_\varphi\frac{\psi(s,n)}{\rho}\right)=-\frac{1}{\rho}\frac{\partial\psi}{\partial s}\mathbf{n}+\frac{1}{\rho}\frac{\partial\psi}{\partial n}\mathbf{s} \tag{39}$$

That is

$$v_n=-\frac{1}{\rho}\frac{\partial\psi}{\partial s},\ v_s=\frac{1}{\rho}\frac{\partial\psi}{\partial n} \tag{40}$$

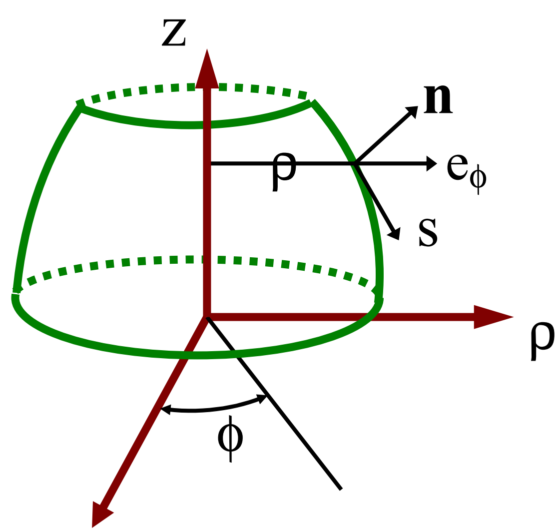

Figure 5. Schematics of intrinsic coordinate systems.

The vorticity is now given as

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = -\mathbf{e}_\varphi \left[\frac{\partial}{\partial n}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial n} \right) + \frac{\partial}{\partial s}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial s} \right) \right] = -\frac{1}{\rho} E^2 \psi \tag{41}$$

where

$$E^2 = \rho \left[\frac{\partial}{\partial n}\left(\frac{1}{\rho}\frac{\partial}{\partial n} \right) + \frac{\partial}{\partial s}\left(\frac{1}{\rho}\frac{\partial}{\partial s} \right) \right]. \tag{42}$$

Note that in terms of ρ and z, E^2 is given by (27) as

$$E^2 = \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}. \tag{43}$$

Plane Stagnation Flows

Consider a steady plane stagnation flow shown in Figure 1. For steady plane flow the Navier-Stokes equation reduces to

$$\frac{\partial\left(\nabla^2\psi,\psi\right)}{\partial(x,y)}=\nu\nabla^4\psi \tag{1}$$

or

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi=\nu\nabla^4\psi \tag{2}$$

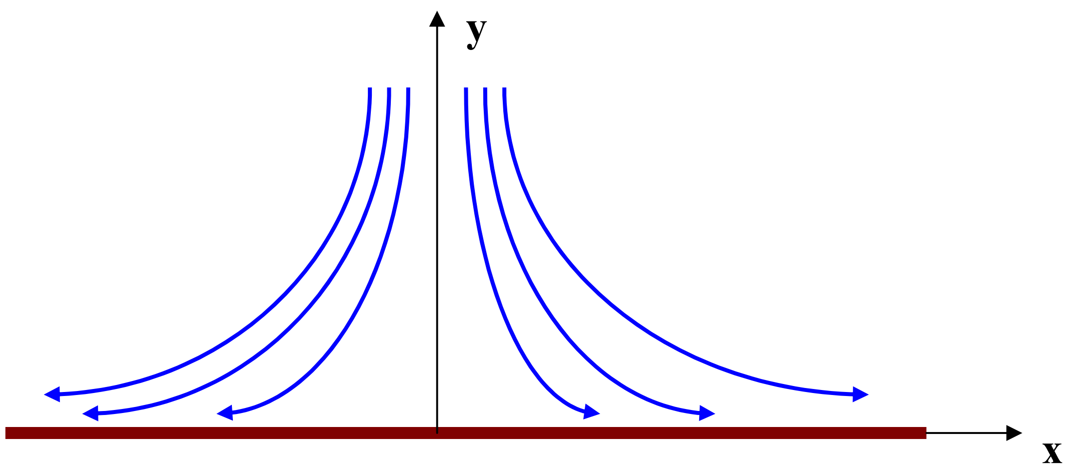

Figure 1. Schematics of plane stagnation flow.

Potential stagnation plane flow is described by

$$U=ax\,,\ V=-ay\,,\ \psi=axy\,. \tag{3}$$

We look for a solution of the form

$$\psi=xf\left(y\right). \tag{4}$$

Then

$$u=xf'\,,\qquad v=f\,,\qquad \nabla^2\psi=xf'' \tag{5}$$

Using (4) and (5), equation (2) becomes

$$xf'f'' - ff'''x = \nu f^{(4)}x \tag{6}$$

or

$$f'f'' - ff''' = \nu f^{(4)}. \tag{7}$$

Integrating (7) we find

$$f'^2 - ff'' = \nu f''' + c \tag{8}$$

The boundary conditions are:

$$\text{At } y = 0, \qquad u = 0, \ v = 0 \tag{9}$$

$$\text{At large y,} \qquad \text{Equations in (3) holds.} \tag{10}$$

That is

$$f(0) = f'(0) = 0, \tag{11}$$

$$\text{As } y \to \infty \qquad f \to ay \tag{12}$$

Using (12) we find

$$c = a^2 \tag{13}$$

Thus

$$f'^2 - ff'' = \nu f''' + a^2 \tag{14}$$

Introducing a change of variable (Schlichting,1960)

$$\eta = \sqrt{\frac{a}{\nu}}y, \ f = \sqrt{a\nu}\varphi(\eta), \tag{15}$$

Equation (14) may be restated as

$$\varphi''' + \varphi\varphi'' - \varphi'^2 + 1 = 0 \tag{16}$$

subject to boundary conditions

$$\phi(0) = \phi'(0) = 0, \tag{17}$$

$$\text{as } \eta \to \infty \qquad \varphi' = 1 \tag{18}$$

Graphical representation of the numerical solution is shown in Figure 2. Additional details of the solution are discussed by Schlichting (1960). Accordingly,

$$\text{At } \eta = 2.4, \quad \varphi' = 0.99, \quad \frac{u}{U} = \phi'(\eta) \tag{19}$$

Hence, the boundary layer thickness is given by

$$\delta = 2.4\sqrt{\frac{\nu}{a}} \tag{20}$$

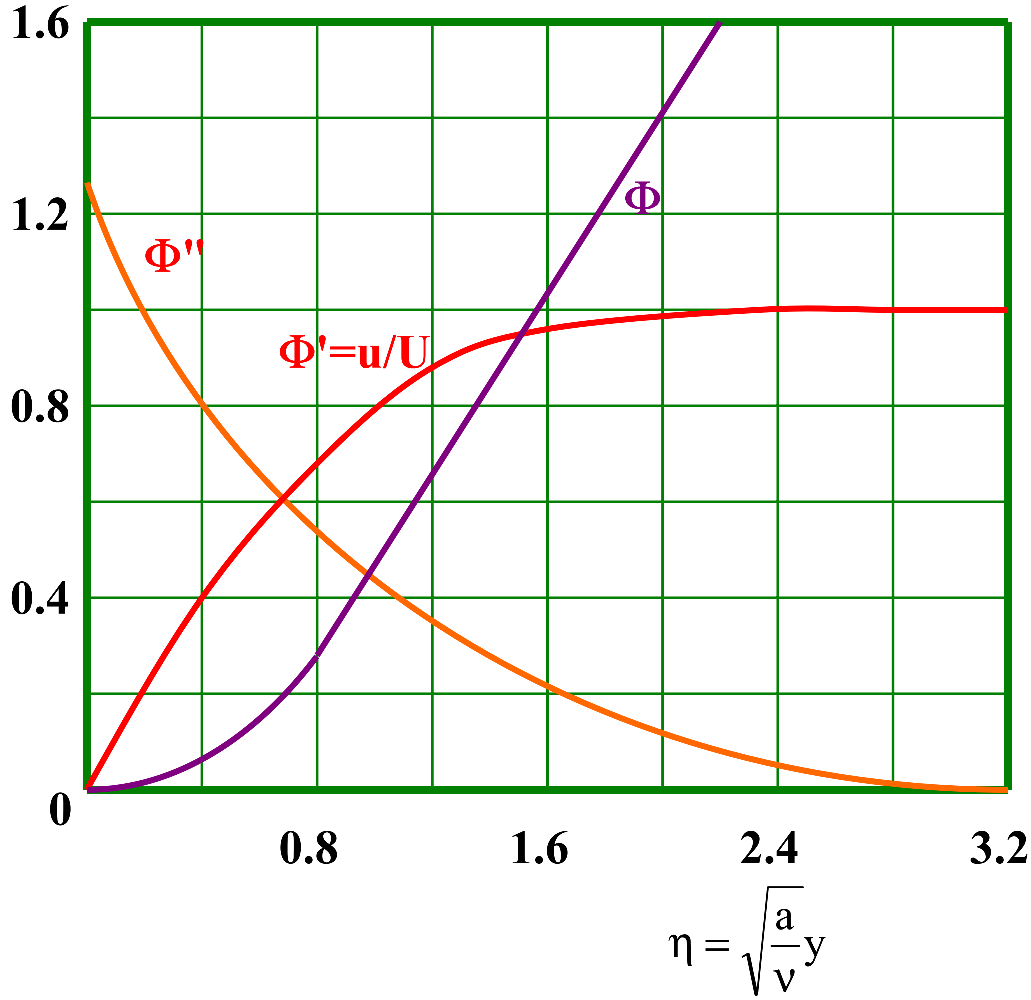

Figure 2. Schematics of plane stagnation flow solutions.

Axisymmetric Stagnation Flow

Potential axisymmetric stagnation flow is described as

$$V_r = ar\,,\ V_z = -2az\,,\ \psi = -ar^2 z\,. \tag{21}$$

The steady state Navier-Stokes equation for axisymmetric flows is given as

$$-\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\left(E^2\psi\right)+\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\left(E^2\psi\right)+\frac{2}{r^2}\frac{\partial\psi}{\partial z}E^2\psi = \nu E^4\psi\,, \tag{22}$$

where

$$v_z = \frac{1}{r}\frac{\partial\psi}{\partial r}\,,\ v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}\,, \tag{23}$$

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\,. \tag{24}$$

Similar to the plane flow case we look for a solution of the form

$$\psi = -r^2 f(z), \tag{25}$$

with

$$v_z = -2f\,,\ v_r = rf'\,,\ E^2\psi = -r^2 f''\,. \tag{26}$$

Using (25) and (26), Equation (22) reduces to

$$rf'\left(-2rf''\right) - 2f\left(-r^2 f'''\right) - 2f'\left(-r^2 f''\right) = -\nu r^2 f^{(4)} \tag{27}$$

or

$$2ff''' + \nu f'''' = 0 \tag{28}$$

The boundary conditions are

$$v_r = v_z = 0 \ \text{ at } \ z = 0\,;\ \psi = -ar^2 z \ \text{ as } \ z \to \infty \tag{29}$$

or

$$f(0) = f'(0) = 0\,,\ f'(\infty) = a \ \text{ or } \ \psi = -ar^2 z \ \text{ as } \ z \to \infty \tag{30}$$

Integrating (28) we find

$$f'^2 - 2ff'' = \nu f''' + c_1 \tag{31}$$

For equation (31) to be valid at large z, $c_1 = a^2$

Introducing a change of variables

$$\xi = \sqrt{\frac{a}{\nu}} z, \quad f = \sqrt{a\nu}\,\phi(\xi) \tag{32}$$

Equation (31) becomes

$$\phi''' + 2\phi\phi'' - \phi'^2 + 1 = 0 \tag{33}$$

subject to

$$\phi(0) = \phi'(0) = 0, \quad \phi'(\infty) = 1 \tag{34}$$

The numerical solutions of the flow filed are very similar to the plane stagnation flow case shown in Figure 2 with a slightly fuller velocity profile. The details of the numerical solution are discussed by Schlichting (1960).