ME 637 **Midterm Exam**

1. (30 Points) Consider the turbulent flow of an incompressible fluid. Estimate the order of magnitude of the following quantities in terms of u, λ, and Λ:

$$\frac{\partial u_i'}{\partial x_i} = 0 = \frac{\partial \omega_i'}{\partial x_i}$$

a) $\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} = \frac{u^2}{\lambda^2} = \frac{v^2}{\eta^2}$

b) $\overline{\frac{\partial \omega_i'}{\partial x_j}\frac{\partial^2 u_j'}{\partial x_i \partial x_k}} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\left(\overline{\omega_i'\frac{\partial u_j'}{\partial x_k}}\right) = \frac{u^2}{\lambda^2 \Lambda^2}$

$\Lambda \eta^2 = \lambda^3$

c) $\overline{\omega_k' \frac{\partial u_k'}{\partial x_i}\omega_i'} = u^3/\lambda^3$

d) $\overline{\frac{\partial \omega_i'}{\partial x_k}\frac{\partial \omega_j'}{\partial x_k}\frac{\partial \omega_i'}{\partial x_m}\frac{\partial \omega_j'}{\partial x_m}} = \frac{u^4}{\lambda^4 \eta^4} = \frac{v^4}{\eta^4 \eta^4} = \frac{v^4}{\eta^8} = \frac{u^4 \Lambda^2}{\lambda^{10}}$

e) $\overline{\frac{\partial u_j'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\omega_i'} = u^3/\lambda^3$

f) $\overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial \omega_i'}{\partial x_k}} = \frac{\partial^2 \overline{u_k'\omega_i'}}{\partial x_i \partial x_k} = \frac{u^2}{\Lambda^3}$

2. (30 Points) The fluctuation velocity in the St Lawrence River is about 1 m/s and the average depth is 20 m, and average width is 1 km. The kinematic viscosity and density of water are $\nu = 10^{-6}\,\mathrm{m^2/s}$ $\rho = 1000\,\mathrm{kg/m^3}$.

a. **Estimate the energy dissipation rate in the river, and the power loss for 1 km length of the river.**
b. **Estimate the Kolmogorov length, time and velocity scales.**
c. **Estimate the Taylor microscale in the river.**

a) $\varepsilon = \frac{u^3}{\Lambda} = \frac{1^3}{20} = \frac{1}{20}\ \mathrm{m^2/s^3}$

$P = m \cdot \varepsilon = (1000 \times 1000 \times 20)\,1000 \cdot \frac{1}{20} = 10^9$ Watts

$= 1000$ MW

b) $\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} = \left(\frac{(10^{-6})^3}{1/20}\right)^{1/4} = 67\ \mu\mathrm{m}$

$v = (\nu\varepsilon)^{1/4} = 1.45$ cm/s

$\tau = (\nu/\varepsilon)^{1/2} = 0.0045$ sec

c) $\varepsilon = \nu \frac{u^2}{\lambda^2} = \nu\, \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}} = \frac{u^3}{\Lambda}$, $\lambda^2 = \frac{\nu\Lambda}{u}$

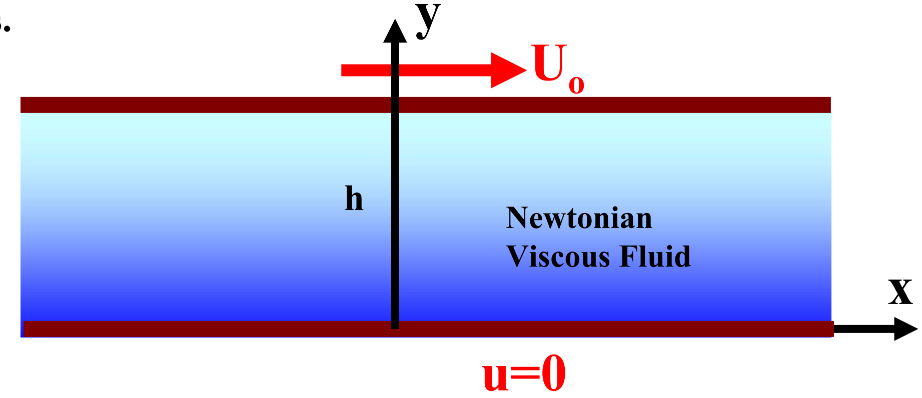
$\lambda^3 = \Lambda\,\eta^2 \Rightarrow \lambda = 0.0045\,\mathrm{m} = 4.5\,\mathrm{mm}$

3. **(40 Point) Consider that case of a viscous Newtonian fluid between two long parallel walls. The upper wall is set suddenly in motion at time zero as shown in the figure. The fluid is initially at rest and there is no pressure gradient**

a. **State the unsteady momentum and continuity equations for parallel the flow shown in it simplest form.**

b. **State the boundary and initial conditions.**

c. **Find the velocity field.**



$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

B.C. At $y=0$ $u=0$

$y=h$ $u=U_0$, $t=0$ $u=0$

$$u = v + U_0 y/h$$

$$\boxed{\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial y^2}}$$

B.C. At $y=0$ $v=0$

$y=h$ $v=0$

At $t=0$ $v = -U_0 y/h$

$$v = Y(y)T(t)$$

$$\frac{1}{\nu}\frac{\dot{T}}{T} = \frac{Y''}{Y} = -\alpha^2$$

$Y'' + \alpha^2 Y = 0$ $\quad Y = A\sin\alpha y + \overset{\nearrow 0}{B}\cos\alpha y$

$\dot{T} + \alpha^2 \nu T = 0$ $\quad T = c e^{-\alpha^2 \nu t}$

$\sin \alpha h = 0 \qquad \alpha h = n\pi \quad , \quad \alpha_n = \frac{n\pi}{h} \qquad n = 1, 2, \ldots$

$Y = A \sin \frac{n\pi y}{h}$

$$v = \sum A_n e^{-\alpha_n^2 \nu t} \sin \frac{n\pi y}{h}$$

At $t=0$

$$-\frac{v_0 y}{h} = \sum A_n \sin \frac{n\pi y}{h}$$

$$A_n = -\frac{v_0}{h} \int_0^h y \sin \frac{n\pi y}{h}\, dy \Big/ \underbrace{\int_0^h \sin^2 \frac{n\pi y}{h}\, dy}_{h/2}$$

$$A_n = \frac{2v_0}{n\pi} (-1)^n$$

$$u = \frac{v_0 y}{h} + \frac{2v_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\left(\frac{n\pi}{h}\right)^2 \nu t} \sin \frac{n\pi y}{h}$$