ME 637 - Particle II

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Creeping Flows

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Outline

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- Navier-Stokes
- Creeping Flow
- Minimum Energy
- Point Force Solution
- General Solution
- Faxen Law
- Nonspherical Particles
- Oblate Ellipsoids
- Porolate Ellipsoids

Creeping Flows

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$$-\nabla p + \mu \nabla^2 \mathbf{v} + \rho f = 0$$

$$\nabla \cdot \mathbf{v} = 0$$

→ $$\nabla^2 p = 0$$

Reciprocity Theorem

Let Two Solution → $(\mathbf{v}, \boldsymbol{\tau})$ $(\mathbf{v}', \boldsymbol{\tau}')$

Then → $$\int_S d\mathbf{S} \cdot \boldsymbol{\tau} \cdot \mathbf{v}' = \int_S d\mathbf{S} \cdot \boldsymbol{\tau}' \cdot \mathbf{v}$$

Proof in Happel and Brenner

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Minimum Energy Dissipation Theorem
Helmhotz

The dissipation rate in creeping flow is less than any other incompressible, continuous motion consistent with the same boundary condition

Proof in Happel and Brenner

Point Force Solution

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Stokes Equation

$$P_{j,i} = \mu \nabla^2 T_{ij} + \delta_{ij}\delta(\boldsymbol{\tau})$$

$$T_{ij,i} = 0$$

Solution-3D

$$T_{ij} = \frac{1}{8\pi\mu r}\left[\delta_{ij} + \frac{r_i r_j}{r^2}\right] \qquad P_i = \frac{r_i}{4\pi r^3}$$

Solution-2D

$$T_{ij} = \frac{1}{4\pi\mu}\left[-\delta_{ij}\ln|r| + \frac{r_i r_j}{r^2}\right] \qquad P_i = \frac{r_i}{2\pi r^2}$$

Point Force Solution

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Minimum Energy Dissipation Theorem

$$v_i = \int_V T_{ij}(\mathbf{r}-\mathbf{r}')\rho f(\mathbf{r}')dV + \int_S [\tau_{jk}(\mathbf{r}')T_{ki}(\mathbf{r}-\mathbf{r}') - R_{ijk}(\mathbf{r}-\mathbf{r}')v_k(\mathbf{r}')]dS_j$$

$$R_{ijk}(\mathbf{r}) = \frac{3r_i r_j r_k}{4\pi r^5}$$

$$p(\mathbf{r}) = \int_V P_i(\mathbf{r}-\mathbf{r}')\rho f_i(\mathbf{r}')dV + \int_S [\tau_{ij}(\mathbf{r}')P_j(\mathbf{r}-\mathbf{r}') + 2\mu v_j(\mathbf{r}')P_{j,i}(\mathbf{r}-\mathbf{r}')]dS_i$$

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In an Unbounded Flow, For a point Force

$$\rho f_i = F_i \delta(\mathbf{r})$$

Stokeslet

$$v_i = F_j T_{ij}(r) \quad \Rightarrow \quad \mathbf{v} = \frac{\mathbf{F}}{8\pi\mu r}\cdot\left[\mathbf{I} + \frac{\mathbf{rr}}{r^2}\right]$$

$$p(r) = F_i P_i(r) \quad \Rightarrow \quad p = \frac{\mathbf{F}\cdot\mathbf{r}}{4\pi r^3}$$

Solution in Spherical Coordinates

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Lamb

$$p = \sum_{n=0}^{\infty} P_n \qquad P_n = (A_n r^n + B_n r^{-n-1})Y_{nm}(\theta,\phi)$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\Rightarrow \quad Y_{mn}(\theta,\phi) = (C_{mn}\cos m\phi + D_{mn}\sin m\phi)P_n^m(\cos\theta)$$

$$m = 0,1,2,\ldots,n \qquad n = 0,1,2,\ldots$$

$$v = \sum_{n=0}^{\infty}\left[\nabla\times(r\xi_n) + \nabla(\Phi_n) + \frac{(n+3)r^2\nabla P_n - 2n\mathbf{r}P_n}{2\mu(n+1)(2n+3)}\right]$$

Spherical Harmonics

Faxen's Law

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For nonuniform velocities

Torque

$$\mathbf{T}_0 = 8\pi\mu R^3\left(\frac{1}{2}\nabla\times\mathbf{v}_\infty|_o - \boldsymbol{\omega}\right)$$

Force

$$\mathbf{F} = 6\pi\mu R\left[\left(\mathbf{v}_\infty|_o - \mathbf{U}\right) + \frac{1}{6}R^2\nabla^2\mathbf{v}_\infty|_o\right]$$

Stresslet

$$\mathbf{S} = \frac{20}{3}\pi R^3\mu\left(1 + \frac{R^2}{10}\nabla^2\right)\mathbf{d}|_o$$

NONSPHERICAL PARTICLES

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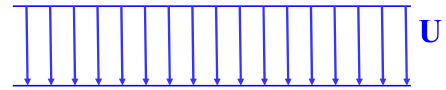
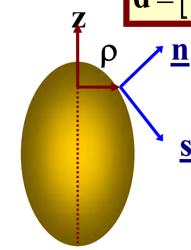
Drag on an Axisymmetric Body



$$\boldsymbol{\tau} = -p\mathbf{I} + 2\mu\mathbf{d}$$

$$\mathbf{d} = \left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right]$$

$$\mathbf{F} = \int_S d\mathbf{S}\cdot\boldsymbol{\tau}$$



$$T = \int_S \mathbf{r}\times\left(d\mathbf{S}\cdot\boldsymbol{\tau}\right)$$

NONSPHERICAL PARTICLES

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Drag on an Axisymmetric Body

$$F_z = \mathbf{F}\cdot\mathbf{e}_z = \int_S d\mathbf{S}\cdot\boldsymbol{\tau}\cdot\mathbf{e}_z = \int_S \mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{e}_z dS$$

$$dS = 2\pi\rho ds$$

$$\mathbf{n}\cdot\boldsymbol{\tau} = -\mathbf{n}p + 2\mu\left(\mathbf{n}\frac{\partial v_n}{\partial n} + \mathbf{s}\frac{\partial v_n}{\partial s}\right) + \mathbf{s}\mu\left(\frac{\partial v_s}{\partial n} - \frac{\partial v_n}{\partial s}\right) = -\mathbf{n}p - 2\mu\nabla\left(\frac{1}{\rho}\frac{\partial\psi}{\partial s}\right) + \mathbf{s}\frac{\mu}{\rho}E^2\psi$$

Drag Force

$$F_z = \mu\pi\int\rho^3\frac{\partial}{\partial n}\left(\frac{E^2\psi}{\rho^2}\right)dS$$

Drag on an Axisymmetric Body

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Flow for a Point Force

$$\psi = \frac{F_z}{8\pi\mu}\frac{\rho^2}{r}$$

$$p = -\frac{F_z}{4\pi}\frac{z}{r^3}$$

Point Particle

$$F_z = 8\pi\mu\lim_{r\to\infty}\frac{r\psi}{\rho^2}$$

$$r^2 = \rho^2 + z^2$$

Fluid not at rest far away

$$F_z = 8\pi\mu\lim_{r\to\infty}\frac{r(\psi - \psi_\infty)}{\rho^2}$$

Oblate Spheroid in a Uniform Flow

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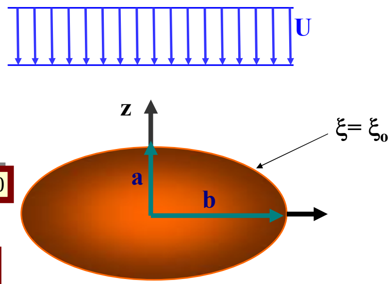
$$\begin{cases} x = c\cosh\xi\sin\theta\cos\phi \\ y = c\cosh\xi\sin\theta\sin\phi \\ z = c\sinh\xi\cos\theta \end{cases}$$



Let

$\lambda = \sinh\xi$ $\quad \infty > \lambda \ge 0$

$\zeta = \cos\theta$ $\quad 1 \ge \zeta \ge -1$

$$\rho = \sqrt{x^2 + y^2} = c\cosh\xi\sin\theta = c\sqrt{\lambda^2+1}\sqrt{1-\zeta^2}$$

$$z = c\lambda\zeta$$

Flow Around an Oblate Spheroid

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Boundary Conditions

$$E^4\psi = 0$$

$\lambda = \lambda_0(\xi = \xi_0)$ → $\psi = 0$

$\lambda = \lambda_0(\xi = \xi_0)$ → $\dfrac{\partial\psi}{\partial\lambda} = 0$

λ or $\xi \to \infty$ → $\psi \to \frac{1}{2}\rho^2 U = \frac{Uc^2}{2}(\lambda^2+1)(1-\zeta^2)$

Flow Around an Oblate Spheroid

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Noting

$$\frac{\partial}{\partial\xi} = \sqrt{\lambda^2+1}\,\frac{\partial}{\partial\lambda} \qquad \frac{\partial}{\partial\theta} = -\sqrt{1-\zeta^2}\,\frac{\partial}{\partial\zeta}$$

$$E^2 = \frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}$$

Leads To

$$E^2 = \frac{1}{c^2(\lambda^2+\zeta^2)}\left[(\lambda^2+1)\frac{\partial^2}{\partial\lambda^2} + (1-\zeta^2)\frac{\partial^2}{\partial\zeta^2}\right]$$

Flow Around an Oblate Spheroid

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Assumed Solution → Boundary Conditions → $\psi = (1-\zeta^2)g(\lambda)$

$$\psi = (1-\zeta^2)\left(-\frac{1}{2}C_1\lambda + \frac{1}{2}C_2\left[\lambda - (\lambda^2+1)\cot^{-1}\lambda\right] + C_3(\lambda^2+1)\right)$$

Solution

$$\psi = \frac{1}{2}U\rho^2\left\{1 - \frac{\dfrac{\lambda}{(\lambda^2+1)} - \left[\dfrac{(\lambda_0^2-1)}{(\lambda_0^2+1)}\right]\cot^{-1}\lambda}{\dfrac{\lambda_0}{(\lambda_0^2+1)} - \left[\dfrac{(\lambda_0^2-1)}{(\lambda_0^2+1)}\right]\cot^{-1}\lambda_0}\right\}$$

$$\lambda_0 = \sinh\xi_0$$

Flow Around an Oblate Spheroid

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Solution for Oblate Spheroids moving in a stationary fluid

$$\psi = -\frac{1}{2}U\rho^2 \frac{\frac{\lambda}{(\lambda^2+1)} - \left[\frac{(\lambda_0^2-1)}{(\lambda_0^2+1)}\right]\cot^{-1}\lambda}{\frac{\lambda_0}{(\lambda_0^2+1)} - \left[\frac{(\lambda_0^2-1)}{(\lambda_0^2+1)}\right]\cot^{-1}\lambda_0}$$

Drag

$$F_z = 8\pi\mu c \lim_{\lambda\to\infty}\frac{\lambda\psi}{\rho^2} \quad\Rightarrow\quad F_z = -\frac{8\pi\mu c U}{\lambda_0 - (\lambda_0^2-1)\cot^{-1}\lambda_0}$$

$$F_z = -6\pi\mu a U K$$

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Flow Around an Oblate Spheroid

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Drag $\Rightarrow$ $F_Z = -6\pi\mu a U K$

$$K = \frac{1}{\left\{\frac{3}{4}\sqrt{\lambda_0^2+1}\left[\lambda_0 - (\lambda_0^2-1)\cot^{-1}\lambda_0\right]\right\}}$$

$$c = \sqrt{a^2-b^2} \qquad \lambda_o = \frac{b}{c} = \frac{1}{\sqrt{(a/b)^2-1}}$$

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Flow Around an Circular Disk

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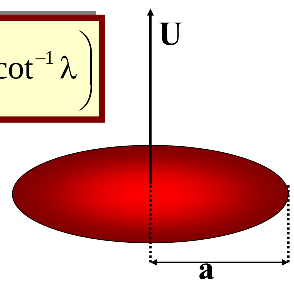
$$\lambda_0 \to 0 \qquad \psi = -\frac{U\rho^3}{\pi}\left(\frac{\lambda}{\lambda^2+1} + \cot^{-1}\lambda\right)$$



Drag $\Rightarrow$ $F_z = -16\mu a U$

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Flow Around a Prolate Spheroid

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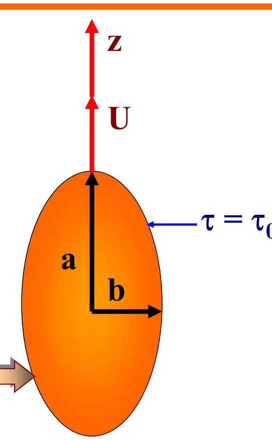
$$\begin{Bmatrix} x = c\sinh\xi\sin\theta\cos\psi \\ y = c\sinh\xi\sin\theta\sin\psi \\ z = c\cosh\xi\cos\theta \end{Bmatrix}$$

Let

$$\zeta = \cos\theta \qquad \tau = \cosh\xi$$

$$\tau = \text{const} \qquad \xi = \text{const} \Rightarrow$$



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Flow Around a Prolate Spheroid

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Solution

$$\psi = -\frac{1}{2}U\rho^2 \frac{\left[\frac{(\tau_0^2+1)}{(\tau_0^2-1)}\right]\coth^{-1}\tau - \left[\frac{\tau}{(\tau^2-1)}\right]}{\left[\frac{(\tau_0^2+1)}{(\tau_0^2-1)}\right]\coth^{-1}\tau_0 - \left[\frac{\tau}{(\tau_0^2-1)}\right]}$$

Drag

$$F_z = -\frac{8\pi\mu c U}{(\tau_0^2+1)\coth^{-1}\tau_0 - \tau_0}$$

$$c = \sqrt{a^2 - b^2}$$

$$\tau_o = \cosh\xi_o = \frac{a}{c} = \frac{1}{\sqrt{1-(b/a)^2}}$$

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Flow Around a Prolate Spheroid

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Drag

$$F_z = -6\pi\mu b U K$$

$$K = \left\{\frac{3}{4}\sqrt{\tau_0^2-1}\left[(\tau_0^2+1)\coth^{-1}\tau_0 - \tau_0\right]\right\}^{-1}$$

Elongated Rode

$$F_z = -\frac{4\pi\mu_a U}{\ln\left(\frac{a}{b}\right) + \ln 2 - \frac{1}{2}}$$

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Concluding Remarks

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Thank you!

Questions?

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