ME 637 - Particle II

Turbulence

Vorticity Equation

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Vorticity Transport in Turbulent Flows

Outline

- ➢ Vorticity Transport in Viscous Flows
- ➢ Mean Flow Vorticity Energy Budget
- ➢ Turbulence Mean Square Vorticity
- ➢ Order of Magnitude Analysis

Vorticity Transport

Navier-Stokes Equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p+\mu\nabla^2\mathbf{u}$$

$$\nabla\cdot\mathbf{u}$$

Vector Identity

$$\mathbf{u}\cdot\nabla\mathbf{u}=(\nabla\times\mathbf{u})\times\mathbf{u}+\nabla\frac{|\mathbf{u}|^2}{2}=\boldsymbol{\omega}\times\mathbf{u}+\nabla\frac{|\mathbf{u}|^2}{2}$$

$$\frac{\partial \mathbf{u}}{\partial t}+\boldsymbol{\omega}\times\mathbf{u}+\nabla\frac{|\mathbf{u}|^2}{2}=-\nabla\frac{p}{\rho}+\nu\nabla^2\mathbf{u}$$

Vorticity Transport Equation

Vorticity ➡ $$\boldsymbol{\omega}=\nabla\times\mathbf{u}$$

Vector Identities

$$\nabla\times\nabla\times\mathbf{u}=\nabla\nabla\cdot\mathbf{u}-\nabla^2\mathbf{u}=-\nabla^2\mathbf{u}$$

Vorticity Transport Eq. ➡ $$\frac{\partial \boldsymbol{\omega}}{\partial t}+\nabla\times(\boldsymbol{\omega}\times\mathbf{u})=\nu\nabla^2\boldsymbol{\omega}$$

Vorticity Transport Equation

Vector Identities

$$\nabla\times(\boldsymbol{\omega}\times\mathbf{u}) = \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \boldsymbol{\omega}(\nabla\cdot\mathbf{u}) - \mathbf{u}(\nabla\cdot\boldsymbol{\omega})$$
$$= \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u}$$

Vorticity Transport Eq.

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} = \mathbf{d}\cdot\boldsymbol{\omega} + \nu\nabla^2\boldsymbol{\omega}$$

Vorticity Transport in Turbulent Flows

Turbulent Flows

$$\mathbf{u} = \mathbf{U} + \mathbf{u}' \qquad U_i = \overline{u_i} \qquad \overline{u_i'} = 0$$

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}' \qquad \boldsymbol{\Omega} = \overline{\boldsymbol{\omega}} \qquad \overline{\boldsymbol{\omega}'} = 0$$

Mean Vorticity Transport Equation

$$\frac{\partial\Omega_i}{\partial t} + U_j\frac{\partial\Omega_i}{\partial x_j} = -\frac{\partial\overline{\omega_i' u_j'}}{\partial x_j} + \frac{\partial\overline{\omega_j' u_i'}}{\partial x_j} + \Omega_j\frac{\partial U_i}{\partial x_j} + \nu\frac{\partial^2\Omega_i}{\partial x_j\partial x_j}$$

Vorticity Transport in Turbulent Flows

Instantaneous Vorticity Transport Equation

$$\frac{\partial\omega_i'}{\partial t} + U_j\frac{\partial\omega_i'}{\partial x_j} = -u_j'\frac{\partial\Omega_i}{\partial x_j} - u_j'\frac{\partial\omega_i'}{\partial x_j} + \Omega_j d_{ij}'$$
$$+ \omega_j' D_{ij} + \omega_j' d_{ij}' + \nu\frac{\partial^2\omega_i'}{\partial x_j\partial x_j} + \frac{\partial\overline{\omega_i' u_j'}}{\partial x_j} - \overline{\omega_j' d_{ij}'}$$

Mean FlowVorticity Energy Eq.

$$\underbrace{\left(\frac{\partial}{\partial t} + \overset{u^3/\Lambda^3}{U_j\frac{\partial}{\partial x_j}}\right)\left(\frac{1}{2}\Omega_i\Omega_i\right)}_{\text{Convective Transport of mean Vorticity}} = \underbrace{-\frac{\partial}{\partial x_j}\left(\Omega_i\overset{u^3/\Lambda^3}{\overline{\omega_i' u_j'}}\right)}_{\text{transport by turbulence velocity-vorticity interaction}} + \underbrace{\overset{u^3/\Lambda^3}{\overline{u_j'\omega_i'}}\frac{\partial\Omega_i}{\partial x_j}}_{\text{gradient production of fluctuating vorticity}}$$
$$+ \underbrace{\overset{u^3/\Lambda^3}{\Omega_i\Omega_j D_{ij}}}_{\text{stretching of vorticity by mean shear flow}} + \underbrace{\Omega_i\overline{\omega_j' d_{ij}'}}_{\text{stretching of vorticity by mean shear flow}}\ u^3/\Lambda^3$$
$$\nu u^2/\Lambda^4 = u^3/\Lambda^3 R_\Lambda$$
$$+\underbrace{\nu\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{2}\Omega_i\Omega_i\right)}_{\text{viscous diffusion}} - \underbrace{\nu\frac{\partial\Omega_i}{\partial x_j}\frac{\partial\Omega_i}{\partial x_j}}_{\text{dissipation of mean vorticity}}$$

Fluctuation Vorticity Energy Eq.

$$\left(\frac{\partial}{\partial t}+\overset{u^3/\Lambda\lambda^2}{U_j\frac{\partial}{\partial x_j}}\right)\left(\frac{1}{2}\overline{\omega_i'\omega_i'}\right)=\underbrace{-\overset{u^3/\Lambda^3}{\overline{u_j'\omega_i'}}\frac{\partial\Omega_i}{\partial x_j}}_{\substack{\text{gradient production}\\ \text{of fluctuating vorticity}}}\underbrace{-\frac{1}{2}\frac{\partial}{\partial x_j}\overset{u^3/\Lambda\lambda^2}{\left(\overline{u_j'\omega_i'\omega_i'}\right)}}_{\substack{\text{diffusion of turbulence}\\ \text{vorticity by turbulence}}}$$

$$+\overset{u^3/\lambda^3}{\underbrace{\overline{\omega_i'\omega_j'd_{ij}'}}_{\substack{\text{production of turbulence vorticity}\\ \text{by turbulent stretching}}}}+\underbrace{\overline{\omega_i'\omega_j'}D_{ij}}_{\substack{\text{production of turbulence vorticity}\\ \text{by mean shear flow stretching}}}\ u^3/\Lambda^2\lambda$$

$$+\overset{u^3/\Lambda^3}{\underbrace{\Omega_j\overline{\omega_i'd_{ij}'}}_{\substack{\text{production by mixed}\\ \text{turbulence-mean flow stretching}}}}\overset{\nu u^2/\lambda^2\Lambda^2\ =\ u^3/\Lambda^3}{\underbrace{+\nu\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{2}\overline{\omega_i'\omega_i'}\right)}_{\text{diffusion by viscosity}}}-\overset{\nu u^2/\lambda^2\eta^2}{\underbrace{\nu\overline{\frac{\partial\omega_i'}{\partial x_j}\frac{\partial\omega_i'}{\partial x_j}}}_{\text{dissipation destruction}}}$$

Order of Terms

$$\boxed{\overline{\omega_i'\omega_i'}\sim\frac{u^2}{\lambda^2}}\qquad\boxed{\overline{u_i'\omega_i'}\sim\frac{u^2}{\Lambda}}\qquad\boxed{\overline{\omega_j'd_{ij}'}\sim\frac{u^2}{\Lambda^2}}$$

$$\boxed{\overline{\omega_i'\omega_j'}\sim\frac{u^2}{\lambda^2}(a\delta_{ij}+b_{ij}\frac{\lambda}{\Lambda}+...)}$$

$$\boxed{\overline{\omega_i'\omega_j'}\frac{\partial U_i}{\partial x_j}=\overline{\omega_i'\omega_j'}D_{ij}\sim\frac{u^2}{\lambda^2}\frac{\lambda}{\Lambda}\frac{u}{\Lambda}=\frac{u^3}{\lambda\Lambda^2}}$$

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Concluding Remarks

- **Vorticity Transport in Viscous Flows**
- **Mean Flow Vorticity Energy Budget**
- **Turbulence Mean Square Vorticity**
- **Order of Magnitude Analysis**

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Thank you!

Questions?