Energy Balance in Turbulent Flow

The Reynolds equation is given as

$$\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{d^2U_i}{\partial x_j\partial x_j}-\frac{\partial}{\partial x_j}\overline{u_i'u_j'}\,. \tag{1}$$

Multiplying (1) by U_i and rearranging terms, one finds

$$\underbrace{\left[\frac{\partial}{\partial t}+U_j\frac{\partial}{\partial x_j}\right]\frac{U_iU_i}{2}}_{\text{convection}}=\underbrace{-\frac{\partial}{\partial x_i}\left[U_i\frac{P}{\rho}+U_j\overline{u_i'u_j'}\right]}_{\text{diffusion by turbulence}}+\underbrace{\nu\frac{\partial^2}{\partial x_j\partial x_j}\frac{U_iU_i}{2}}_{\text{viscous diffusion}}-\underbrace{\nu\frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j}}_{\text{viscous dissipation}}+\underbrace{\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}}_{\text{fluctuation energy production}} \tag{2}$$

where

$$\left[\frac{\partial}{\partial t}+U_j\frac{\partial}{\partial x_j}\right]\frac{U_iU_i}{2}\ = \text{convection},$$

$$-\frac{\partial}{\partial x_i}\left[U_i\frac{P}{\rho}+U_j\overline{u_i'u_j'}\right] = \text{diffusion by turbulence},$$

$$\nu\frac{\partial^2}{\partial x_j\partial x_j}\frac{U_iU_i}{2}\ = \text{viscous diffusion},$$

$$\nu\frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j}\ = \text{direct viscous dissipation},$$

$$\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}\ = \text{fluctuation energy production}.$$

Equation (2) is the statement of balance of mean mechanical energy for the mean motion.

Subtracting (1) from the Navier-Stokes equation, it follows that

$$\frac{\partial u_i'}{\partial t}+U_j\frac{\partial u_i'}{\partial x_j}+u_j'\frac{\partial U_i}{\partial x_j}+u_j'\frac{\partial u_i'}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_j}+\nu\frac{\partial^2u_i'}{\partial x_j\partial x_j}+\frac{\partial\overline{u_i'u_j'}}{\partial x_j} \tag{3}$$

Multiplying (3) by u_i' and taking expected value, we find the equations of balance turbulence energy fluctuation. i.e.,

$$\underbrace{\frac{\partial k}{\partial t}+U_j\frac{\partial k}{\partial x_j}}_{\text{Convection}}=\underbrace{-\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}}_{\text{Production}}\underbrace{-\frac{\partial}{\partial x_j}\left[\frac{\overline{u_j'p'}}{\rho}+\frac{\overline{u_i'u_i'u_j'}}{2}\right]}_{\text{turbulent Diffusion}}+\underbrace{\nu\frac{\partial^2 k}{\partial x_j\partial x_j}}_{\substack{\text{Viscous}\\ \text{Dissipation}}}-\underbrace{\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}}_{\text{Dissipation}} \tag{4}$$

where $k=\frac{\overline{u_i'u_i'}}{2}$ is the fluctuation kinetic energy and

$$\frac{\partial k}{\partial t}+U_j\frac{\partial k}{\partial x_j} = \text{convection}$$

$$\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j} = \text{production}$$

$$\frac{\partial}{\partial x_j}\left[\frac{\overline{u_j'p'}}{\rho}+\frac{\overline{u_i'u_i'u_j'}}{2}\right] = \text{turbulent diffusion}$$

$$\nu\frac{\partial^2 k}{\partial x_j\partial x_j} = \text{viscous diffusion}$$

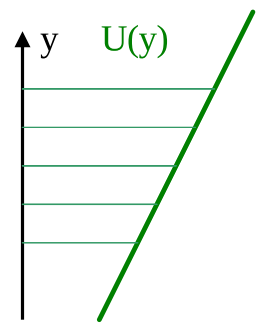
$$\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} = \text{dissipation}$$

Energy Equation in a Pure Shear Flow

The exact (unclosed) energy equation is given by

$$\frac{D}{Dt}\frac{\overline{u_i' u_i'}}{2} = -\frac{\partial}{\partial x_j}\left[\overline{u_j'\left(\frac{p'}{\rho}+\frac{u_i' u_i'}{2}\right)}\right] - \overline{u_i' u_j'}\frac{\partial U_i}{\partial x_j} - \varepsilon + \overline{\frac{p'}{\rho}\frac{\partial u_j'}{\partial x_j}}$$

Note that the last term is zero.



For a pure shear flow,

$$\mathbf{U} = (U_1(y),0,0) \text{ and } \frac{\partial}{\partial x} = \frac{\partial}{\partial z} = \mathbf{U}\cdot\nabla = 0 .$$

The energy equation then becomes

$$0 = -\frac{\partial}{\partial y}\left[\overline{u_2'\left(\frac{p'}{\rho}+\frac{u_i' u_i'}{2}\right)}\right] - \overline{u_1' u_2'}\frac{\partial U_1}{\partial y} - \varepsilon + \overline{\frac{p'}{\rho}\frac{\partial u_j'}{\partial x_j'}} .$$

The energy equations for $\frac{1}{2}\overline{u_1'^2}$, $\frac{1}{2}\overline{u_2'^2}$, and $\frac{1}{2}\overline{u_3'^2}$ are given as

$(\frac{1}{2}\overline{u_1'^2})$: $$0 = -\frac{\partial}{\partial y}\left(\overline{u_2'\frac{u_1' u_1'}{2}}\right) - \overline{u_1' u_2'}\frac{\partial U_1}{\partial y} - \frac{1}{3}\varepsilon + \overline{\frac{p'}{\rho}\frac{\partial u_1'}{\partial x}}$$

$(\frac{1}{2}\overline{u_2'^2})$: $$0 = -\frac{\partial}{\partial y}\left[\overline{u_2'\left(\frac{p'}{\rho}+\frac{u_2' u_2'}{2}\right)}\right] - \frac{1}{3}\varepsilon + \overline{\frac{p'}{\rho}\frac{\partial u_2'}{\partial y}}$$

$(\frac{1}{2}\overline{u_3'^2})$: $$0 = -\frac{\partial}{\partial y}\left[\overline{u_2'\left(\frac{u_3' u_3'}{2}\right)}\right] - \frac{1}{3}\varepsilon + \overline{\frac{p'}{\rho}\frac{\partial u_3'}{\partial z}}$$

It is observed that the entire production is for $\frac{\overline{u_1'^2}}{2}$ and there is no direct production of $\overline{u_2'^2}$ and $\overline{u_3'^2}$. Therefore, u_2' and u_3' receive their energy from the pressure-velocity interaction terms. That is, $\overline{p'\frac{\partial u_2'}{\partial y}}$ and $\overline{p'\frac{\partial u_3'}{\partial z}}$ must be positive and $\overline{p'\frac{\partial u_1'}{\partial x}}$ must be negative. In most flows, $\frac{\overline{u_1'^2}}{2}$ is twice as large as $\frac{\overline{u_2'^2}}{2}$ and $\overline{u_3'^2}$.