Correlation, Spectrum, and Scales

***Definition*: Correlation Tensor (Two-Point)**

Consider a turbulent flow field as shown in Figure 1. Here $\mathbf{u}_1$ and $\mathbf{u}_1$ are the components of the velocity vectors and $\mathbf{r} = \mathbf{x_1} - \mathbf{x_2}$ is the distance between the two points. The two-point correlation tensor is defines as

$$Q_{ij}(\mathbf{x}, \mathbf{x_1}) = \overline{u_i'(\mathbf{x}) u_j'(\mathbf{x_1})}$$

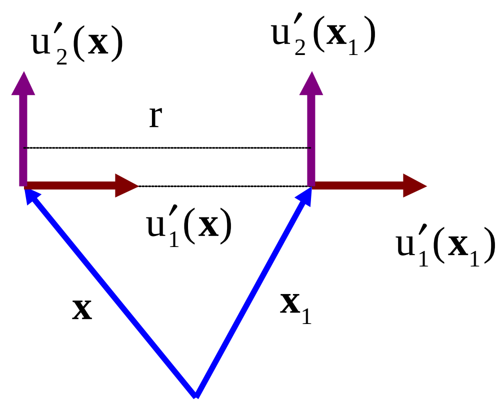

Figure 1. Geometric features of two-point correlations in a turbulent flow field.

In a homogenous turbulent flow, the correlations (and all the statistics) are independent of the shift of space origin. That is,

$$Q_{ij}(\mathbf{x}, \mathbf{x_1}) = Q_{ij}(\mathbf{r})$$

***Definition*: Longitudinal Correlation Coefficient**

The longitudinal correlation coefficient is defines as

$$f(r) = \frac{Q_{11}}{u_1^2}$$

where

$$Q_{11} = \overline{u_1'(\mathbf{x}) u_1'(\mathbf{x_1})}, \qquad u_1^2 = \overline{u_1'^2(\mathbf{x})} = \overline{u_1'^2(\mathbf{x_1})}$$

Note that f(r) is an even function. That is

$$f(r) = f(-r)$$

A typical longitudinal correlation coefficient is shown in Figure 2.

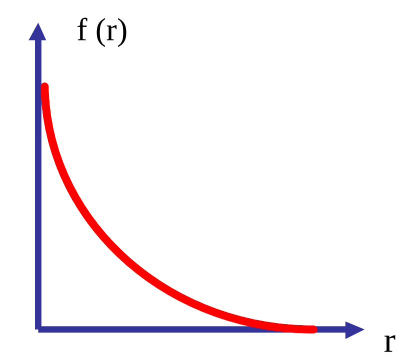

Figure 2. Schematics of longitudinal correlation coefficient.

Definition: **Lateral Correlation Coefficient**

The lateral correlation coefficient is defines as

$$g(r) = \frac{Q_{22}}{u_2^2}, \qquad Q_{22} = \overline{u_2'(\mathbf{x})u_2'(\mathbf{x_1})}$$

The lateral correlation coefficient is also a symmetric Function. That is

$$g(r) = g(-r)$$

A typical lateral correlation coefficient is shown in Figure 3.

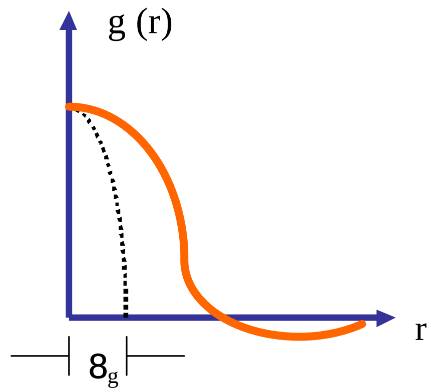

Figure 3. Schematics of lateral correlation function and the corresponding Taylor Scale.

***Definition*: Taylor's Microscales**

The Taylor microscales are defines as

$$\lambda_f^2 = -\frac{2}{f''(0)}, \quad \lambda_g^2 = -\frac{2}{g''(0)},$$

where λ_f and λ_g are, respectively, the Taylor longitudinal and lateral microscales. The microscales may be defined by fitting a parabola to the correlation coefficient curves at $r = 0$. That is,

$$g(r) = 1 + \frac{1}{2!} r^2 g''(0) + \cdots \approx 1 - \frac{r^2}{\lambda_g^2}$$

***Definition*: Integral Scales, Macroscales**

The macroscales of turbulence are defined as

$$\Lambda_f = \int_0^\infty f(r)dr \ = \text{Longitudinal Macroscale}$$

$$\Lambda_g = \int_0^\infty g(r)dr \ = \text{Lateral Macroscale}$$

***Definition*: Eulerian Time Correlation (stationary flows)**

The Eulerian time correlation is defined as

$$R_E(\tau) = \frac{\overline{u_1'(\mathbf{x},t)u_1'(\mathbf{x},t+\tau)}}{u_1^2}$$

The Eulerian time microscale τ_E then is given by

$$\tau_E^2 = -\frac{2}{R_E''(0)}$$

The Eulerian time macroscale (integral scale) T_E is defined as

$$T_E = \int_0^\infty R_E(\tau)d\tau$$

Using the uniform flow and frozen field approximations, the scales may be related. That is

$$\Lambda_f \approx UT_E, \qquad \lambda_f \approx U\tau_E, \qquad f(U\tau) \approx R_E(\tau)$$

$$\frac{\partial}{\partial t} = -U\frac{\partial}{\partial x}$$

***Definition*: Lagrangian Time Correlation**

The Lagrangian velocity correlation is defined as

$$R_L(\tau) = \frac{\overline{v_L'(t)v_L'(t+\tau)}}{\overline{v_L'^2}}$$

where v_L' is the Lagrangian fluctuation velocity. The corresponding Lagrangian time microscale τ_L and the macroscale T_L are given as

$$\tau_L^2 = -\frac{2}{R_L''(0)}$$

and

$$T_L = \int_0^\infty R_L(\tau) d\tau$$

***Definition*: Energy Spectrum Tensor**

Energy spectrum of is defined as the Fourier Transform of the correlation tensor. That is

$$E_{ij}(\mathbf{k}) = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q_{ij}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{dx}$$

$$Q_{ij}(\mathbf{x}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E_{ij}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{dk}$$

***Definition*: One Dimensional Energy Spectrum**

The one dimensional energy spectrum is defined as

$$E_1(k_1) = \frac{u_1^2}{\pi} \int_{-\infty}^{+\infty} f(x_1) e^{-ik_1x_1} dx_1$$

$$u_1^2 f(x_1) = \frac{1}{2} \int_{-\infty}^{+\infty} E_1(k_1) e^{ik_1x_1} dk_1$$

Symmetry of $f(x_1)$ implies that

$$E_1(k_1) = \frac{2u_1^2}{\pi} \int_0^\infty f(x_1) \cos k_1 x_1 dx_1$$

$$u_1^2 f(x_1) = \int_0^\infty E_1(k_1) \cos k_1 x_1 dk_1$$

A typical one dimensional energy spectrum is shown in Figure 4.

Setting x_1 equal to zero, we find

$$u_1^2 = \int_0^\infty E_1(k_1) dk_1 \, .$$

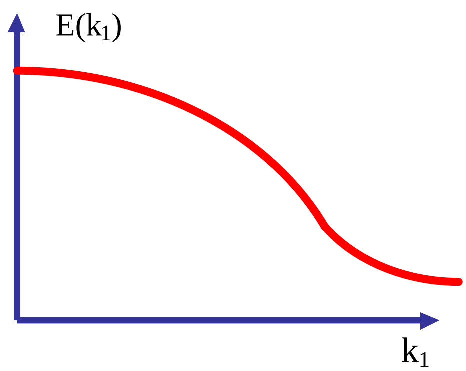

Figure 4. Schematics of one-dimensional energy spectrum.

Also

$$\frac{1}{\lambda_f^2} = -\frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}\bigg|_{x_1=0} = \frac{1}{2u_1^2}\int_0^\infty k_1^2 E(k_1)dk_1$$

***Example*:** The longitudinal correlation may be approximated as

$$f(r) = e^{-\frac{r}{\Lambda_f}}$$

The corresponding one dimensional spectrum is given as

$$E_1(k_1) = \frac{u_1^2}{\pi}\frac{2\Lambda_f}{1+\Lambda_f^2 k_1^2}$$

Estimates for Taylor Microscales

The energy dissipation is given as

$$\varepsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} \tag{1}$$

For isotropic turbulence, it can be shown that

$$\varepsilon = 15\nu \overline{(\frac{\partial u_1'}{\partial x_1})^2} = -15\nu \overline{u_1'^2} f''(0) \tag{2}$$

It then follows that

$$\varepsilon = 30\nu \frac{\overline{u_1'^2}}{\lambda_f^2} = 30\nu \frac{u^2}{\lambda_f^2} = 15\nu \frac{u^2}{\lambda_g^2} \tag{3}$$

Since

$$\lambda_f^2 = \lambda_g^2 \sqrt{2} \tag{4}$$

Using the macroscopic estimate for the dissipation, we find

$$\varepsilon = A\frac{u^3}{\Lambda} = 30\nu \frac{u^2}{\lambda_f^2} \tag{5}$$

Therefore

$$\frac{\lambda_f}{\Lambda} = \sqrt{\frac{30}{A}} R_\Lambda^{-1/2} \tag{6}$$

Therefore,

$$\frac{\lambda_f}{\Lambda} << 1 \qquad \text{since} \quad R_\Lambda = \frac{u\Lambda}{\nu} >> 1 \tag{7}$$

Similarly

$$\frac{\lambda_g}{\Lambda} = \sqrt{\frac{15}{A}} R_\Lambda^{-1/2} \tag{8}$$

It may also be shown that

$$\frac{\lambda_g}{\Lambda} = \frac{15}{A} R_\lambda^{-1}, \qquad R_\lambda = \frac{u\lambda}{\nu} \tag{9}$$

and

$$\frac{\lambda_g}{\eta} = (\frac{225}{A})^{1/4} R_\Lambda^{-1/4} = 15^{1/4} R_\lambda^{-1/2} \tag{10}$$

From Equation (3) it follows that

$$\frac{u}{\lambda_g} = 0.26\sqrt{\frac{\varepsilon}{\nu}} = \frac{0.26}{\tau} \tag{11}$$

where the Kolmogorov time scale is given by

$$\tau = \frac{\eta}{\upsilon} = \sqrt{\frac{\nu}{\varepsilon}} \tag{12}$$

This means that the Taylor microscale is not a characteristic length of the dissipation eddies. It, however, provides a useful artificial length scale for estimating the velocity gradients of the small eddies when macroscopic velocity scale is used for the velocity of the eddies. That is, Equations (1) and (11) imply that

$$\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} = \frac{\varepsilon}{\nu} \sim (\frac{u}{\lambda_g})^2 \tag{13}$$

Other useful estimates are

$$\overline{d_{ij}' d_{ij}'} \sim \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} \sim (\frac{u}{\lambda})^2, \qquad \frac{u^3}{\Lambda} \sim \nu\frac{u^2}{\lambda^2}, \qquad \nu \sim \frac{u\lambda^2}{\Lambda} \tag{14}$$

Also

$$\frac{\lambda}{\Lambda} \sim R_\Lambda^{-1/2} \sim R_\lambda^{-1}, \qquad \frac{\eta}{\Lambda} \sim R_\Lambda^{-3/4} \sim R_\lambda^{-3/2}, \qquad \frac{\eta}{\lambda} \sim R_\Lambda^{-1/4} \sim R_\lambda^{-1/2} \tag{15}$$

and

$$\eta^2\Lambda = \lambda^3 \tag{16}$$