ME 637 Solution to HW SET 3

1) (Problem 1.3, Tennekes and Lumley) Large eddies in turbulent flows have a length scale ℓ and a time scale $t(\ell) = \ell / u$. The smallest eddies have a length scale of η, a velocity scale of ***L***, and time scale τ. Estimate the characteristic velocity ***L***(r) and characteristic time t(r) of eddies of size r, where r is in the range of $\eta < r < \ell$. (Note that in this range ***L***(r) and t(r) are determined by ε and r.) Show that your results agrees with the know results at $r = \eta$ and $r = \ell$. Find an express for the energy spectrum of turbulence, $E(\kappa) = \dfrac{v^2(\kappa)}{\kappa}$.

Solutions:

Scales	**Length**	**Velocity**	**Time**
Large Eddies	ℓ	$v(\ell) = u$	$t(\ell) = \ell / u$
Smallest Eddies	$\eta = \left(\dfrac{\nu^3}{\varepsilon}\right)^{1/4}$	$v(\eta) = (\nu\varepsilon)^{1/4}$	$\tau(\eta) = \left(\dfrac{\nu}{\varepsilon}\right)^{1/2}$
For eddies of size r $\eta < r < \ell$	r	$v(r) = (r\varepsilon)^{1/3}$	$t(r) = \left(\dfrac{r^2}{\varepsilon}\right)^{1/3}$

For eddies of size r with $\eta < r < \ell$, the length scale is r and $\varepsilon = \dfrac{u^3}{\ell}$, and the corresponding velocity and time scales are $v(r) = (r\varepsilon)^{1/3}$ and $t(r) = \left(\dfrac{r^2}{\varepsilon}\right)^{1/3}$.

Large Scale limit, $r = \ell$, and $u = (\ell\varepsilon)^{1/3} = u$, $t = (\ell^2 / \varepsilon)^{1/3} \Rightarrow t = \ell / u$

Kolmogorov scale limit, $r = \eta$, and

$$u = (\varepsilon\eta)^{1/3} = \left(\varepsilon(\nu^3/\varepsilon)^{1/4}\right)^{1/3} \Rightarrow u = (\nu\varepsilon)^{1/4}$$

$$t = (\eta^2/\varepsilon)^{1/3} = \left(\frac{(\nu^3/\varepsilon)^{1/2}}{\varepsilon}\right)^{1/3} \Rightarrow t = (\nu/\varepsilon)^{1/2}$$

The limiting results are the same as the known results at $r = \eta$ and $r = \ell$.

Energy spectrum of turbulence is given as $E(\kappa) = \dfrac{v^2(\kappa)}{\kappa}$, where $\kappa = 1/r$ is the wave number. The velocity scale is $v(\kappa) = (\varepsilon r)^{1/3} = \left(\dfrac{\varepsilon}{\kappa}\right)^{1/3}$.

Thus, $E(\kappa)=\dfrac{v^2(\kappa)}{\kappa}=\dfrac{1}{\kappa}\left(\dfrac{\varepsilon}{\kappa}\right)^{2/3}=\varepsilon^{2/3}\kappa^{-5/3}$.

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2) (Problem 3.1, Tennekes and Lumley) Estimate the characteristic velocity of eddies whose size is equal to the Taylor microscale λ. (See problem 1) Show that eddies of this size dissipates little energy.

Solutions:

From Problem 1, it follows that for an eddy of size λ, the characteristic velocity is $v(\lambda)=(\lambda\varepsilon)^{1/3}$. The corresponding dissipation is

$\varepsilon=\nu\overline{\dfrac{\partial u_i'}{\partial x_j}\dfrac{\partial u_i'}{\partial x_j}}$ and dissipation of eddies of eddies of size λ

$$\varepsilon_\lambda=\nu\frac{v(\lambda)^2}{\lambda^2}=\nu\frac{(\lambda\varepsilon)^{2/3}}{\lambda^2}=\nu\frac{\varepsilon^{2/3}}{\lambda^{4/3}}$$

or $\dfrac{\varepsilon_\lambda}{\varepsilon}=\nu\dfrac{(\lambda\varepsilon)^{2/3}}{\lambda^2}=\dfrac{\nu}{\varepsilon^{1/3}\lambda^{4/3}}=\left(\dfrac{\nu^3}{\varepsilon\lambda^4}\right)^{1/3}=\left(\dfrac{\nu^3}{\nu u^2\lambda^2}\right)^{1/3}=\left(\dfrac{\nu^2}{u^2\lambda^2}\right)^{1/3}=\dfrac{1}{R_\lambda^{2/3}}=\dfrac{1}{R_\Lambda^{1/3}}$

Thus eddies of size λ dissipate little energy. In thus derivations, we used $\varepsilon=\nu\dfrac{u^2}{\lambda^2}$ and also $R_\lambda\sim R_\Lambda^2$.

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3) Derive the energy equation for the Burger model

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}$$

Assume $u=U+u'$. Discuss the meaning of the terms in the energy equation.

Solutions:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}$$

Using Reynolds decomposition to Burger equation $u=U+u'$ after averaging we find

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=\nu\frac{\partial^2 U}{\partial x^2}-\frac{1}{2}\frac{\partial\overline{u'^2}}{\partial x}$$

Multiplying with U and rearranging terms we find:

$$\underbrace{\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{U^2}{2}}_{\text{Convection}}=\underbrace{-\frac{\partial}{\partial x}\left(U\frac{\overline{u'^2}}{2}\right)}_{\text{Turbulent Diffusion}}+\underbrace{\nu\frac{\partial^2}{\partial x^2}\left(\frac{U^2}{2}\right)}_{\text{Vicous Diffusion}}\underbrace{-\nu\left(\frac{\partial U}{\partial x}\right)^2}_{\text{Dissipation}}+\underbrace{\frac{\overline{u'^2}}{2}\frac{\partial U}{\partial x}}_{\substack{\text{Energy Transfer to}\\ \text{Fluctuating Motion}}}$$

Subtracting the equation for the mean from Burger Equation, we find the equation for the fluctuating velocity field. That is

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}=\nu\frac{\partial^2 u'}{\partial x^2}-u'\frac{\partial U}{\partial x}-u'\frac{\partial u'}{\partial x}+\frac{1}{2}\frac{\partial\overline{u'^2}}{\partial x}$$

Multiplying by u' and averaging we fine

$$\underbrace{\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\overline{u'^2}}{2}}_{\text{Convection}}=\underbrace{\frac{\nu}{2}\frac{\partial^2\overline{u'^2}}{\partial x^2}}_{\text{Viscous Diffusion}}\underbrace{-\overline{u'^2}\frac{\partial U}{\partial x}}_{\text{Production}}\underbrace{-\frac{1}{3}\frac{\partial\overline{u'^3}}{\partial x}}_{\text{TurbulentDiffusion}}\underbrace{-\nu\overline{\left(\frac{\partial u'}{\partial x}\right)^2}}_{\text{Dissipation}}$$

4) Consider a turbulent flow between two parallel plates. Derive the expression for the velocity in the viscous sublayer and in the log region. Assume the two solution should match at $y^+=10$. Assuming that the log profile is valid up to the channel centerline, find the expression for the friction coefficient

$$C_f=\frac{\tau_o}{\frac{1}{2}\rho U_c^2}=2\left(\frac{u^*}{U_c}\right)^2$$

Solutions:

For the log region, the logarithmic velocity profile is given as

$$U^+=\frac{1}{\kappa}\ln y^+ + B$$

For viscous sublayer

$$U^+=y^+$$

Assuming the two velocity profile match at $y^+=10$, therefore,

$$10=\frac{1}{\kappa}\ln 10+B$$

For $\kappa=0.4$, we find $B=10-2.5\ln 10$, or $B=4.24$. Thus

$$U^+=\frac{u}{u*}=2.5\ln\frac{yu*}{\nu}+4.24$$

Assuming that the log profile is valid up to the channel centerline at y=h, it follows that,

$$\frac{U_c}{u*} = 2.5 \ln \frac{hu*}{\nu} + 4.24$$

Friction coefficient is defined as

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_c^2} = 2\left(\frac{u^*}{U_c}\right)^2$$

Hence

$$C_f = \frac{2}{\left(2.5\ln\frac{hu*}{\nu} + 4.24\right)^2} = \frac{2}{\left(2.5\ln\left(\frac{hU_c}{\nu}\frac{u*}{U_c}\right) + 4.24\right)^2}$$

Since $\frac{u^*}{U_c} = \sqrt{\frac{C_f}{2}}$, and $\text{Re} = \frac{hU_c}{\nu}$, it follows that

$$C_f = \frac{2}{\left(2.5\ln\left(\text{Re}\sqrt{\frac{C_f}{2}}\right) + 4.24\right)^2}.$$