Several Random Variables

Given a probability experiment $\Im:(S,F,P)$, a random vector $\mathbf{X}(\xi)=(X_1(\xi),X_2(\xi),...,X_n(\xi))$ is defined as a mapping of the probability space unto a point of the n-dimensional Euclidean space R^n. That is $\mathbf{X}(\xi)$ is defined by a certain rule for every $\xi \in S$.

Joint Distribution Function

The joint distribution of n random variables X_1, X_2, … , X_n is defined as

$$F_{\mathbf{X}}(x_1,...,x_n)=P\{X_1 \le x_1,...,X_n \le x_n\}.$$

Joint Density Function

The joint density function is defined by

$$f_{\mathbf{X}}(x_1,...,x_n)=\frac{\partial^n F_{\mathbf{X}}(x_1,...,x_n)}{\partial x_1...\partial x_n}.$$

Properties:

1. $F_{\mathbf{X}}(\infty,\infty,...,\infty)=\int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty} f_{\mathbf{X}}(x_1,...,x_n)dx_1...dx_n=1$.
2. $P\{X_1(\xi),...,X_n(\xi)\in D\}=\int_D...\int f(x_1,x_2,...,x_n)dx_1...dx_n$.

Independent Random Variables

The random variables X_1, X_2, …, X_n are said to be independent if the events $\{X_1 \le x_1\},\ldots,\{X_n \le x_n\}$ are independent for any $x_1,\ldots,x_n$.

If X_1, X_2, …, X_n are independent random variables, then

$$F_{\mathbf{X}}(x_1,x_2,...,x_n)=F_1(x_1)F_2(x_2)...F_n(x_n),$$

and

$$f_x(x_1,x_2,...,x_n)=f_1(x_1)f_2(x_2)...f_n(x_n).$$

Expected Value

The expected value is defined as

$$E\{g(X_1,\ldots,X_n)\} = \int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} g(x_1,\ldots,x_n) f_{\mathbf{X}}(x_1,\ldots,x_n) dx_1\ldots dx_n \ .$$

Covariance

The covariance of two random variables X_i and X_j is defined as

$$c_{ij} = E\{(X_i - \eta_i)(X_j - \eta_j)\} = E\{X_i X_j\} - \eta_i \eta_j \ .$$

where

$$\eta_i = E\{X_i\}.$$

Characteristic Function

The joint characteristic function is defined as

$$\Phi_{\mathbf{X}}(\omega_1,\ldots,\omega_n) = E\{e^{i(\omega_1 X_1 + \ldots + \omega_n X_n)}\} = E\{e^{i\omega\cdot X}\}.$$

The characteristic and the density function of Fourier transform pair, i.e.

$$\Phi_{\mathbf{X}}(\omega) = \int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} e^{i\omega\cdot\mathbf{x}} f_{\mathbf{X}}(\mathbf{x}) dx_1\ldots dx_n \ .$$

and

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} e^{-i\omega\cdot\mathbf{x}} \Phi_{\mathbf{x}}(\omega) d\omega_1\ldots d\omega_n \ .$$

If X_1, X_2,…, X_n are independent random variables, then

$$\Phi(\omega_1,\ldots,\omega_n) = \Phi_1(\omega_1)\ldots\Phi_n(\omega_n).$$