Clarkson University

Transformation of Several Random Variables

A Function of Several Random Variables

Let Z be a function of X and Y. i.e.,

$$Z(\xi) = g[X(\xi), Y(\xi)].$$

Given the joint density of X and Y, $f_{XY}(x, y)$, we would like to find the density or distribution of Z.

Procedure

By definition,

$$F_Z(z) = P\{Z(\xi) \le z\} = P\{g(x, y) \le z\}.$$

Let D_Z be the region of the xy-plane such that $g(x, y) \le z$. Then

$$F_Z(z) = P\{(X, Y) \in D_Z\} = \iint_{D_Z} f_{XY}(x, y) dxdy.$$

Two Functions of Two Random Variables

Let

$$Z = g(X, Y),$$

$$W = h(X, Y).$$

Given $f_{XY}(x, y)$ find $F_{ZW}(z, w)$ or $f_{ZW}(z, w)$.

Procedure

By definition

$$F_{ZW}(z, w) = P\{Z \le z \cap W \le w\} = P\{g(X, Y) \le z \cap h(X, Y) \le w\}.$$

Let D_{zw} be the region in xy-plane such that

$$g(x, y) \le z \text{ and } h(x, y) \le w.$$

Then,

$$F_{ZW} = P\{(X,Y) \in D_{ZW}\} = \iint_{D_{ZW}} f_{XY}(x,y)dxdy .$$

Definition: Uncorrelated RV

The random variables X_i are uncorrelated if

$$E\{X_i X_j\} = E\{X_i\}E\{X_j\}, \text{ or } E\{(X_i - \eta_i)(X_j - \eta_j)\} = 0 ,\ i \neq j$$

It then follows that

$$\sigma^2_{X_1 + \ldots + X_n} = \sigma^2_{X_1} + \sigma^2_{X_2} + \ldots + \sigma^2_{X_n} .$$

Definition: Orthogonal RandomVariables

The random variables X_i are orthogonal if

$$E\{X_i X_j\} = 0 \text{ for } i \neq j .$$

Theorem: If the random variables X_i are independent, then they are uncorrelated.

Example: Jointly Normal Random Variables.

Two random variables X and Y are jointly normal if

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left\{ -\frac{1}{2(1-r^2)} \left[\frac{(x-\eta_1)^2}{\sigma_1^2} - \frac{2r(x-\eta_1)(x-\eta_2)}{\sigma_1\sigma_2} + \frac{(y-\eta_2)^2}{\sigma_2^2} \right] \right\}$$

where

$$\eta_1 = E\{X\},\ \eta_2 = E\{Y\},\ \sigma_1^2 = E\{(X-\eta_1)^2\},\ \sigma_2^2 = E\{(Y-\eta_2)^2\},$$

and correlation coefficient is

$$r = \frac{E\{(X-\eta_1)(Y-\eta_2)\}}{\sigma_1\sigma_2} .$$

Two random variables X and Y are uncorrelated (and independent in this case) if $r = 0$.