Poisson Process

Review of Poisson Random Variable

$X(\xi)$ is a Poisson random variable if

$$P\{X(\xi)=k\}=e^{-a}\frac{a^k}{k!},$$

or

$$f_X(x)=\sum_{k=0}^{\infty}e^{-a}\frac{a^k}{k!}\delta(x-k).$$

It then follows that

$$E\{X\}=a, \qquad E\{X^2\}=a^2+a, \qquad \sigma_x^2=a.$$

Consider a probability experiment of placing points at random on a lie. Define $n(t_1,t_2)$ as the number of points in an interval (t_1,t_2). Then $X(t)=n(0,t)$ is a Poisson random variable if

i) $P\{n(t_1,t_2)=k\}=e^{-\lambda t}\dfrac{(\lambda t)^k}{k!}$, $t=t_2-t_1$.

ii) If the intervals (t_1,t_2) and (t_3,t_4) are non-overlapping, then the random variables $n(t_1,t_2)$ and $n(t_3,t_4)$ are independent.

That is, $X(t)$ is a Poisson process with parameter λt.

Thus,

$$P\{X(t)=k\}=e^{-\lambda t}\frac{(\lambda t)^k}{k!},$$

$$E\{X(t)\}=\lambda t, \; E\{X^2(t)\}=\lambda^2t^2+\lambda t.$$

To obtain the autocorrelation of $X(t)$, assume that $t_2>t_1$ and consider

$$E\{X(t_1)[X(t_2)-X(t_1)]\}=E\{X(t_1)X(t_2)\}-E\{X^2(t_1)\}.$$

Noting that the intervals $(0,t_1)$ and (t_1,t_2) are non-overlapping, it follows that

$$E\{X(t_1)\}E\{X(t_2)-X(t_1)\}=R(t_1,t_2)-\left(\lambda^2 t_1^2+\lambda t_1\right)$$

or

$$\lambda t_1[\lambda(t_2-t_1)]=R(t_1,t_2)-\lambda^2 t_1^2-\lambda t_1 .$$

Thus,

$$R(t_1,t_2)=\lambda^2 t_1 t_2+\lambda t_1,\ t_2\geq t_1 .$$

Similarly,

$$R(t_1,t_2)=\lambda^2 t_1 t_2+\lambda t_2,\ t_1\geq t_2 .$$

Therefore,

$$R(t_1,t_2)=\lambda^2 t_1 t_2+\lambda\min(t_1,t_2).$$

Non-Uniform Case

If the points placed on the line have non-uniform density $\lambda(t)$, then λt must be replaced by $\int_0^t \lambda(\tau)d\tau$. In this case,

$$P\{X(t)=k\}=e^{-\int_0^t \lambda(\tau)d\tau}\frac{\left(\int_0^t \lambda(\tau)d\tau\right)^k}{k!},$$

$$E\{X(t)\}=\int_0^t \lambda(\tau)d\tau ,$$

$$R(t_1,t_2)=\int_0^{t_1}\lambda(\tau_1)d\tau_1\int_0^{t_2}\lambda(\tau_2)d\tau_2+\int_0^{\min(t_1,t_2)}\lambda(\tau)dz .$$

Weiner Process (Brownian Motion)

$W(t)$ is a Wiener Process if

i) It is a normal process with $f(w;t)=\frac{1}{\sqrt{2\pi\alpha t}}e^{-\frac{w^2}{2\alpha t}}$ and

$$F(w;t)=\frac{1}{2}+erf\,\frac{w}{\sqrt{\alpha t}} .$$

ii) $W(t)$ is an independent increment process, i.e. $W(t_2)-W(t_1)$ is independent of $W(t_4)-W(t_3)$ if the intervals (t_1,t_2) and (t_3,t_4) are non-overlapping.

iii) $W(0)=0$.

The statistics of the Wiener Process are given as

$$E\{W(t)\}=0,$$

$$E\{W^2(t)\}=\alpha t,$$

$$R(t_1,t_2)=E\{W(t_1)W(t_2)\}=\begin{Bmatrix}\alpha t_2 & t_1 \ge t_2 \\ \alpha t_1 & t_2 \ge t_1\end{Bmatrix}=\alpha \min(t_1,t_2).$$

White Noise Process

The process $n(t)=\frac{dW(t)}{dt}$ is referred to as the White Noise process. The statistics of $n(t)$ are given by

$$E\{n(t)\}=0,\ R(t_1,t_2)=\alpha\delta(t_1-t_2).$$

Clarkson University

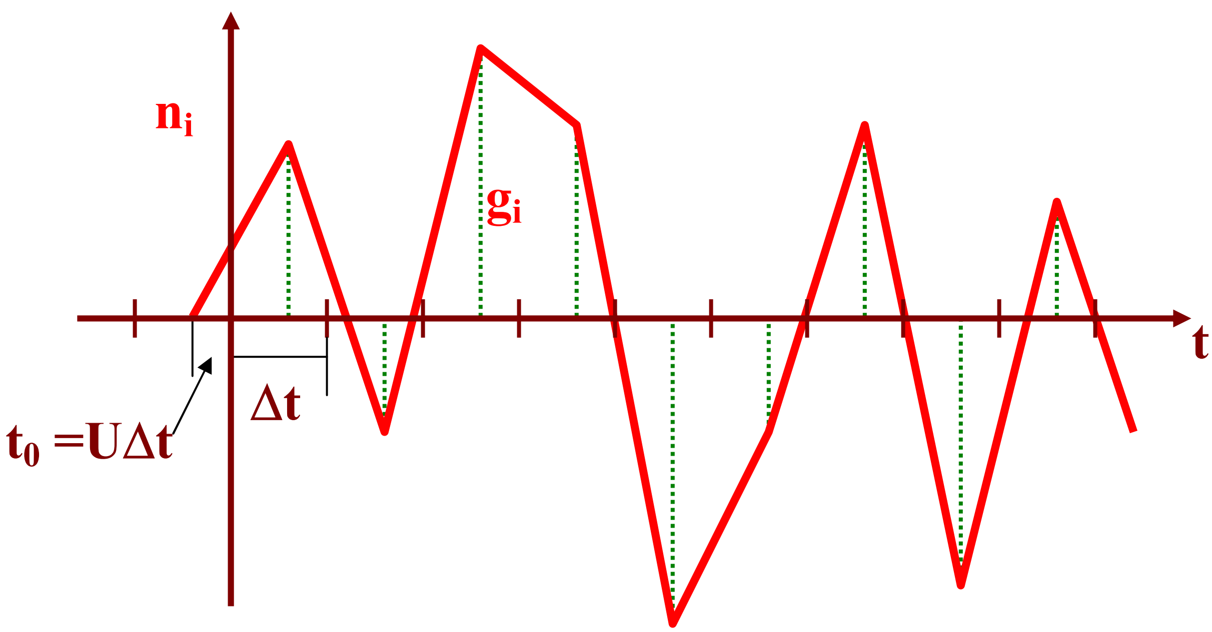
Numerical Simulation of White Noise Processes

An approximately white noise process can be generated using the following step-by-step procedure:

i) For a duration of T (about 20 s) select a small time step Δt (about 0.01 to 0.05 s). The duration is then divided into $m = \frac{T}{\Delta t}$ (about 400 to 2000) subintervals.

ii) Generate $m+1$ zero-mean unit-variance normally distributed random numbers $G_1, \ldots, G_{m+1}$. Multiply these random numbers by $\left(\frac{2\pi S_0}{\Delta t}\right)^{\frac{1}{2}}$ where S_0 is the constant power spectrum of the white noise. Evaluate $g_i = \left(\frac{2\pi S_0}{\Delta t}\right)^{\frac{1}{2}} G_i$.

iii) The white noise process then is given by $n(t_0 + i\Delta t) = g_i$, $i = 1,2,\ldots,m+1$, and $n(t_0) = 0$, and n varies linearly over each subinterval. Here, t_0 is a random variable with uniform density over the subinterval $(-\Delta t, 0)$.



Numerical procedure for simulating a white nose Process.

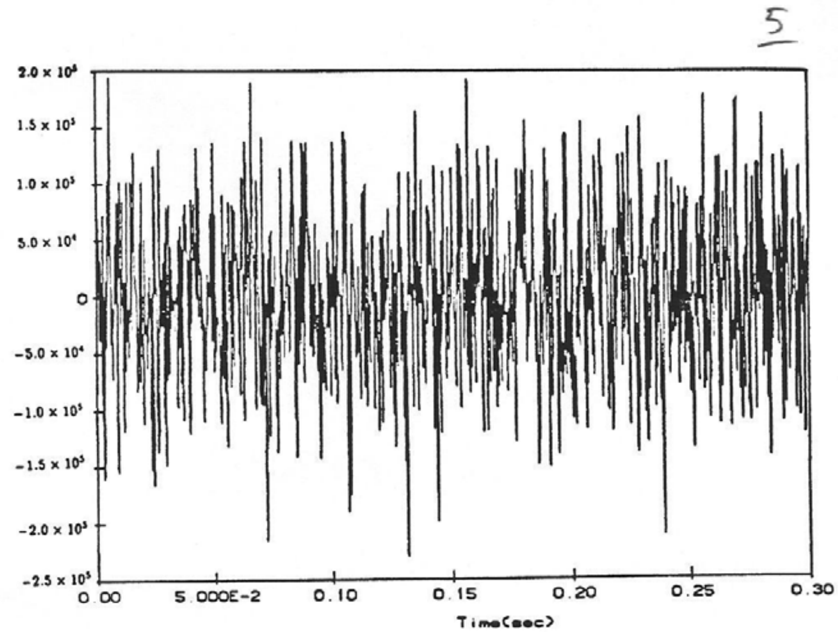


Fig. 1. A sample white noise process vs time, $d_p = 0.05\ \mu m$.

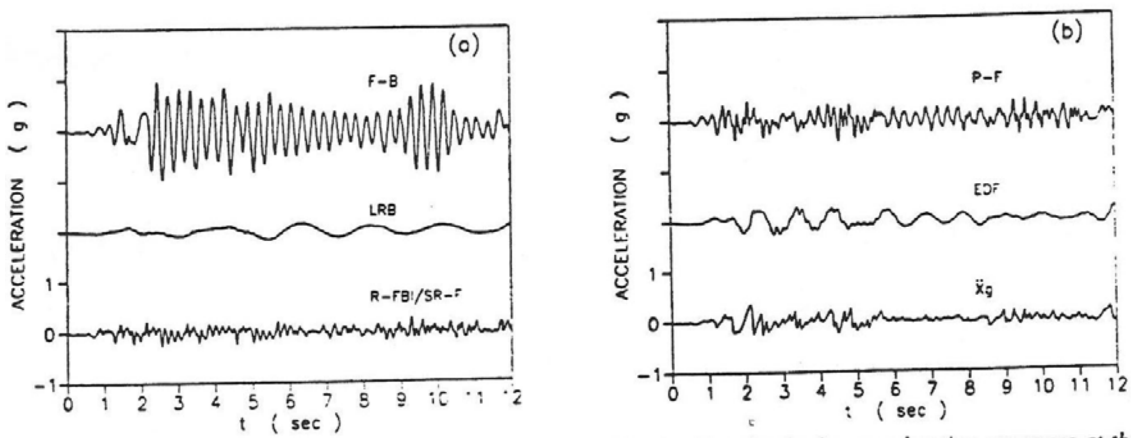


Fig. 3. Sample absolute acceleration responses at the top of structure for El Centro 1940 earthquake

Transformation Form Pair of Uniform Random Variable to Gaussian

Transform pairs of uniform random numbers to pairs of unit variance zero mean Gaussian random numbers can be done using the following transformations:

$$G_1 = \sqrt{-2\ln U_1}\cos 2\pi U_2 ,$$
$$G_2 = \sqrt{-2\ln U_1}\sin 2\pi U_2 .$$

Normal Processes

A stochastic process $X(t)$ is said to be normal if $X(t_1)$, $X(t_2),\ldots,$ $X(t_n)$ are jointly normal for any n and any t_1, $t_2,\ldots,$ t_n. The statistics of a normal process are fully determined in terms of its mean $\eta(t)$ and its autocorrelation $R(t_1,t_2)$ (or autocovariance $C(t_1,t_2)$). The first order density is given as

$$f(x;t)=\frac{1}{\sqrt{2\pi C(t,t)}}e^{\frac{-[x-\eta(t)]^2}{2C(t,t)}}$$

The nth order joint density is given by

$$f(x_1,\ldots,x_n;t_1,\ldots,t_n)=\frac{1}{(2\pi)^{\frac{n}{2}}|\Lambda|^{\frac{1}{2}}}exp\left\{-\frac{1}{2}\sum_i\sum_j\Lambda_{ij}^{-1}(x_i-\eta(t_i))(x_j-\eta(t_j))\right\}$$

where $\mathbf{\Lambda}$ is the matrix of covariance function defined as

$$\mathbf{\Lambda}=\left[C(t_i,t_j)\right] \text{ and } |\Lambda|=\det|\Lambda|.$$

Note: Linear combinations of normal processes (or random variables) are also normal processes.