Clarkson University

Analysis of Linear Systems

Deterministic Systems

Consider a linear system

$$LY(t) = X(t), \tag{1}$$

where L is a deterministic linear differential operator. Also let

$$Y(0) = \frac{dY(0)}{dt} = ... = 0. \tag{2}$$

Suppose $h(t)$ is the impulse response of the linear system. That is,

$$Lh(t) = \delta(t), \tag{3}$$

with

$$h(0) = ... = 0. \tag{4}$$

Formally we may write

$$h(t) = L_t^{-1}\delta(t). \tag{5}$$

Similarly,

$$Y(t) = L_t^{-1}X(t) = L_t^{-1}\int_{-\infty}^{+\infty} X(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{+\infty} X(\tau)L_t^{-1}\delta(t-\tau)d\tau \tag{6}$$

or

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau\ . \tag{7}$$

Noting that $X(t) = h(t) = 0$ for $t < 0$, we find

$$Y(t) = \int_0^t h(t-\tau)X(\tau)d\tau\ . \tag{8}$$

This equation is the basis for the analysis of deterministic (or random) linear systems. When $X(t)$ act for $-\infty < t < +\infty$, then equation (7) must be used. Alternatively,

$$Y(t) = \int_{-\infty}^{+\infty} h(\tau) X(t-\tau) d\tau \, . \qquad (9)$$

Random Linear Systems

Consider a linear system which is identified by its impulse response $h(t)$ or its system function $H(\omega)$ (or $H(i\omega)$). Note that

$$H(\omega) = H(i\omega) = \int_{-\infty}^{+\infty} h(t) e^{-i\omega t} dt \, . \qquad (10)$$

Stationary Response Analysis

Suppose $X(t)$ is a stationary input and $Y(t)$ is a stationary response, then

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau) X(\tau) d\tau = \int_{-\infty}^{+\infty} h(\tau) X(t-\tau) d\tau \, . \qquad (11)$$

Note that $h(t) = 0$ for $t < 0$. Thus Equation (11) is equivalent to

$$Y(t) = \int_{-\infty}^{t} h(t-\tau) X(\tau) d\tau = \int_{0}^{+\infty} h(\tau) X(t-\tau) d\tau \, . \qquad (12)$$

Mean of Y(t)

Taking expected value of (11), we find

$$E\{Y(t)\} = \int_{-\infty}^{+\infty} h(\tau) E\{X(t-\varepsilon)\} d\tau = \eta_X \int_{-\infty}^{+\infty} h(\tau) d\tau = \eta_X H(0). \qquad (13)$$

Autocorrelation and Cross-Correlation

Multiplying (11) by $X(t-\tau)$ and taking expected value we find

$$E\{Y(t) X(t-\tau)\} = \int_{-\infty}^{+\infty} E\{X(t-\alpha) X(t-\tau)\} h(\alpha) d\alpha \qquad (14)$$

or

$$R_{YX}(\tau) = \int_{-\infty}^{+\infty} R_{XX}(\tau-\alpha) h(\alpha) d\alpha = R_{XX}(\tau) * h(\tau). \qquad (15)$$

That is, the cross-correlation of Y and X is the convolution of $R_{XX}(\tau)$ and $h(\tau)$.

Multiplying (11) by $Y(t+\tau)$ and taking expected value, the result is

$$E\{Y(t+\tau)Y(\tau)\} = \int_{-\infty}^{+\infty} E\{Y(t+\tau)X(t-\alpha)\}h(\alpha)d\tau\,,$$

or

$$R_{YY}(\tau) = \int_{-\infty}^{+\infty} R_{YX}(\tau+\alpha)h(\alpha)d\alpha = \int_{-\infty}^{+\infty} R_{YX}(\tau-\alpha)h(-\alpha)d\alpha\,. \qquad (16)$$

That is,

$$R_{YY}(\tau) = R_{YX}(\tau) * h(-\tau). \qquad (17)$$

Similarly, one may show

$$R_{XY}(\tau) = R_{XX}(\tau) * h(-\tau), \qquad (18)$$

and

$$R_{YY}(\tau) = R_{XY}(\tau) * h(\tau). \qquad (19)$$

Thus,

$$R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau). \qquad (20)$$

Clearly, stationary input produces stationary input for a linear system.

System Identification

For a white noise input with $R_{XX}(\tau) = \delta(\tau)$, Equation (15) yields

$$R_{YX}(\tau) = h(\tau).$$

Thus, evaluating

$$E\{Y(t+\tau)X(t)\} \approx \frac{1}{T}\int_0^t Y(t+\tau)X(t)dt \approx R_{YX}(\tau),$$

gives the impulse response.

Power Spectrum

Recalling that the Fourier transform of the convolution of two functions is the product of their Fourier transforms, from (15) it follows that

$$S_{YX}(\omega) = S_{XX}(\omega) H(\omega) \tag{21}$$

Similarly, from (7), (18), (19), and (20) one finds

$$S_{YY}(\omega) = S_{YX}(\omega) H^*(\omega), \tag{22}$$

$$S_{XY}(\omega) = S_{XX}(\omega) H^*(\omega), \tag{23}$$

$$S_{YY}(\omega) = S_{XY}(\omega) H^*(\omega), \tag{24}$$

and

$$S_{YY}(\omega) = S_{XX}(\omega) |H(\omega)|^2 . \tag{25}$$

In these equations, $H(\omega)$ is the system function defined by Equation (10) and $H^*(\omega)$ is its complex conjugate. i.e.,

$$H^*(\omega) = \int_{-\infty}^{+\infty} h(-\tau) e^{-i\omega t} d\tau = \int_{-\infty}^{+\infty} h(\tau) e^{i\omega t} d\tau . \tag{26}$$

Furthermore, Impulse response function and the system functions are Fourier pair. That is,

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} H(\omega) d\omega . \tag{27}$$

Spectral Relationships

Given a linear differential equation with constant coefficients

$$a_n \frac{d^n Y}{dt^n} + a_{n-1} \frac{d^{n-1} Y}{dt^{n-1}} + \ldots + a_0 Y = X(t,),$$

the system function is given as

$$H(\omega) = \frac{1}{a_n (i\omega)^n + \ldots + a_0} .$$

More generally, taking Fourier transform

$$\overline{Y}(\omega) = H(\omega) \overline{X}(\omega),$$

where $H(\omega)$ is the system function. The expected value of $Y, E\{Y\}$, then is given as

$$E\{Y\} = H(0)E\{X\} = \frac{1}{a_0}E\{X\},$$

and the power spectrum of the response then is given by

$$S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2 .$$

Example: Langevin's Equation (Brownian Motion)

The equation of motion of a Brownian particle is given as

$$\frac{dV}{dt} + \beta V = n , \qquad E\{n\} = 0 , \qquad S_{nn}(w) = \alpha .$$

The power spectrum of V is then given as

$$S_{VV}(\omega) = |H(\omega)|^2 S_{nn}(\omega), \qquad H(\omega) = \frac{1}{i\omega + \beta}, \qquad |H(\omega)|^2 = \frac{1}{\omega^2 + \beta^2}$$

Therefore,

$$S_{VV}(w) = \frac{\alpha}{\omega^2 + \beta^2} .$$

The corresponding autocorrelation of V becomes

$$R_{VV}(\tau) = \frac{\alpha}{2\beta}e^{-\beta|\tau|},$$

and

$$E\{V^2\} = \frac{\alpha}{2\beta}, \; E\{V\} = 0 .$$

Motions of small particles in a turbulent simple shear flow field under microgravity condition

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Motions of small rigid spheres in a turbulent flow field in the presence of a uniform mean shear and in the absence of gravitational effect are studied. The particle equation of motion, which includes the Stokes drag and the Saffman lift force effects, is treated as a stochastic differential equation. The spectral method is used and analytical expressions relating the components of particle response statistics to that of the flow field are developed. The particle spectral intensities, autocorrelation functions, and mean-square velocities, as well as particle diffusivities for different particle relaxation times and mean shear rates are evaluated. It is shown that the presence of a mean shear field enhances the particle diffusivity in the transverse direction. Experimental observations of particle mass diffusivity greater than that of the fluid particle may then be explained by this shear-induced enhancement.

I. INTRODUCTION

Understanding the dispersive action of turbulence in free shear flows (jets and wakes) and wall bounded shear fields (pipes and channels) has attracted considerable attention in the past three decades. Extensive reviews of literature on diffusion of finite size particles in turbulent flows were

ity tensor. Tavoularis and Corrsin[34,35] used analytical and experimental methods for evaluating the heat diffusivity tensor in a turbulent simple shear flow field in the presence of a

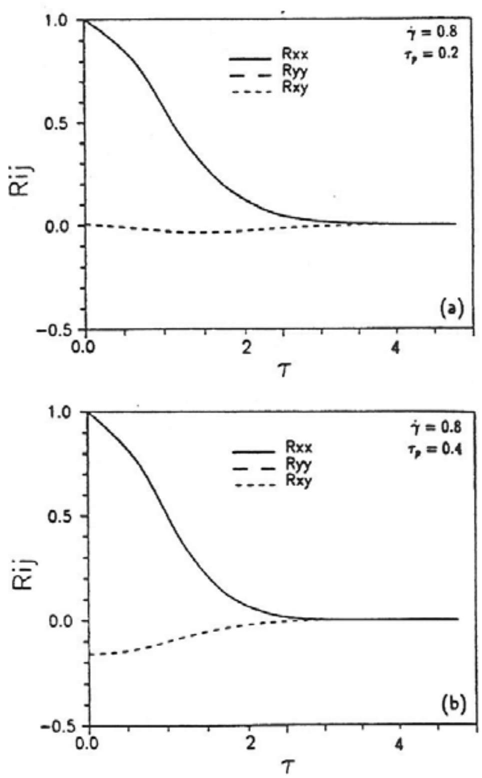

FIG. 2. Variations of normalized particle autocorrelation.

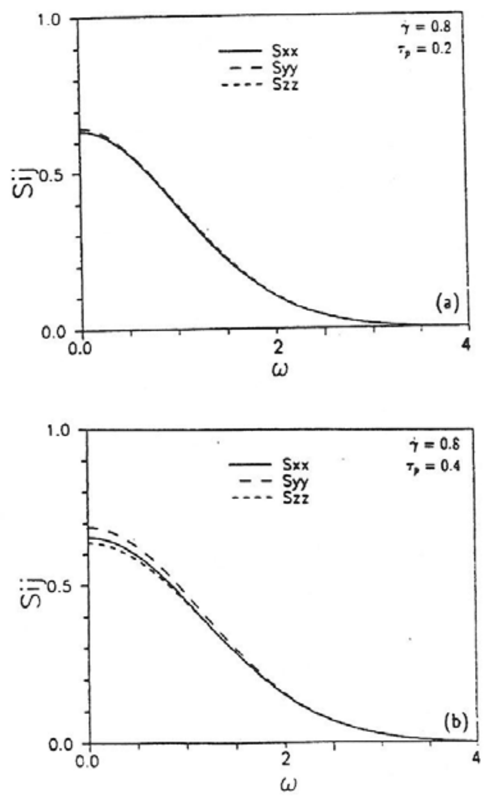

FIG. 1. Variations of particle power spectra.

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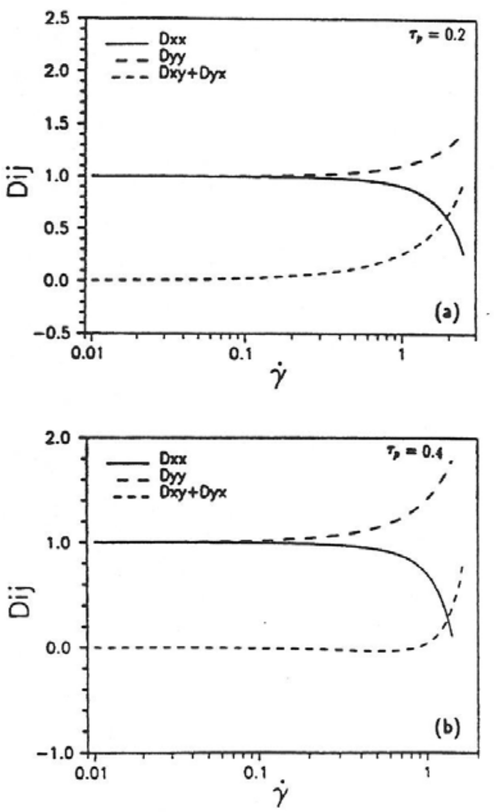

FIG. 6. Variations of particle diffusivity components with shear rate.

components ($D^p_{xy} + D^p_{yx}$) is negative and its amplitude increases with τ_p.

Variations of components of the particle mass diffusivity with shear rate for different particle relaxation times are shown in Fig. 13. The axial component of diffusivity has a

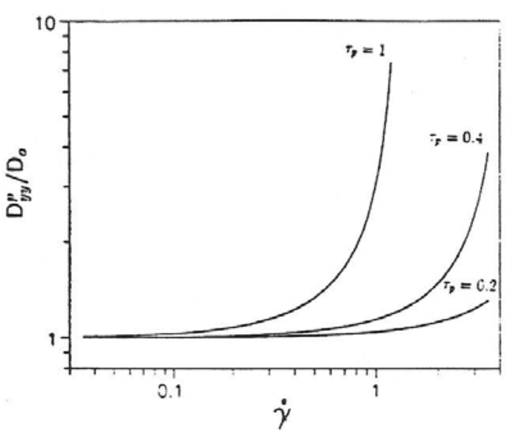

FIG. 7. Variations of normalized mass diffusivity across the shear field with shear rate for different particle relaxation times.

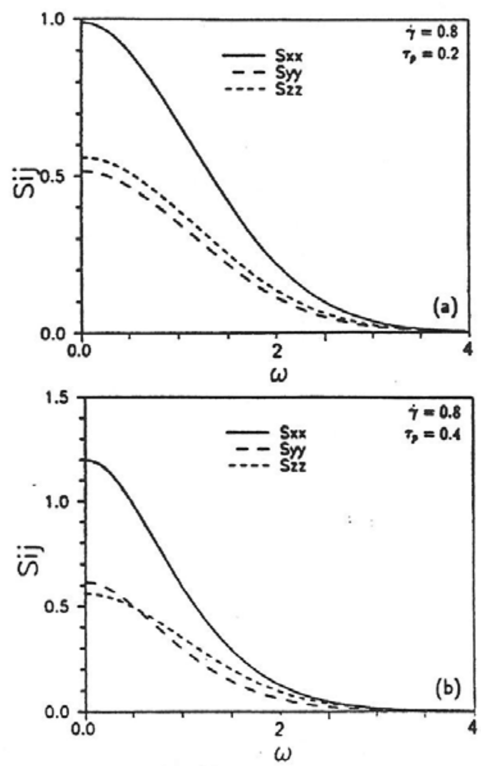

FIG. 8. Variations of particle power spectra.

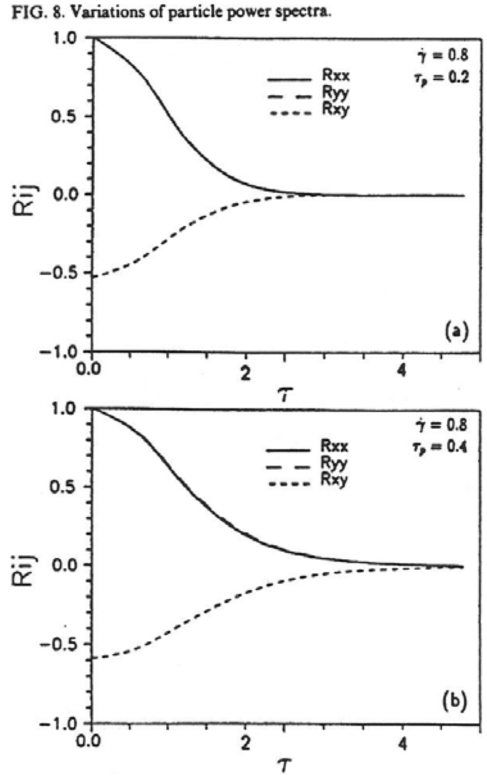

FIG. 9. Variations of normalized particle autocorrelation.

Wind Effects on Building

$$\bar{V}(t) = V_0 \quad \text{(18a)}$$

$$\overline{\xi(t)} = 0 \quad \text{(18b)}$$

An empirical expression for the (two-sided) power spectrum of the fluctuating part of the along wind velocity, which was used by Yang and Lin (1981a,b), is given as

$$S_{\xi\xi}(\omega) = \frac{S(\omega)}{V_0^2} = \frac{2k_0}{|\omega|}\frac{\Omega^2}{(1+\Omega^2)^{4/3}} \quad \text{(19a)}$$

$$\Omega = \frac{600\omega}{\pi V_0} \quad \text{(19b)}$$

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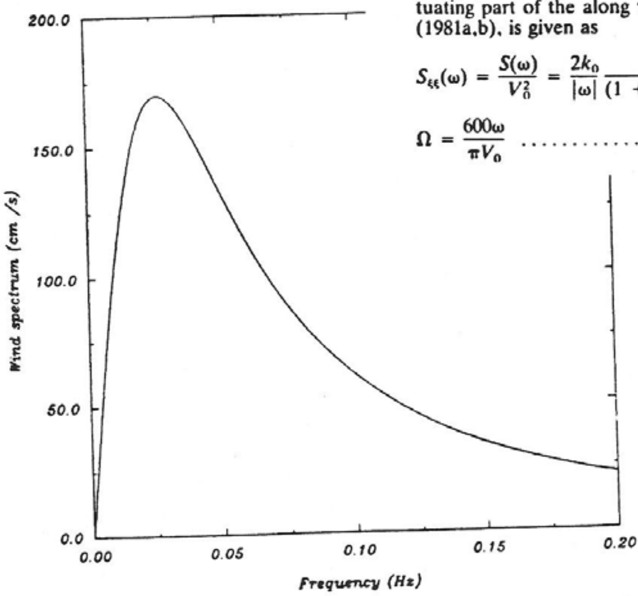

FIG. 4. Power Spectrum of Fluctuating Part of Wind Velocity for V_0 = 42 m/s

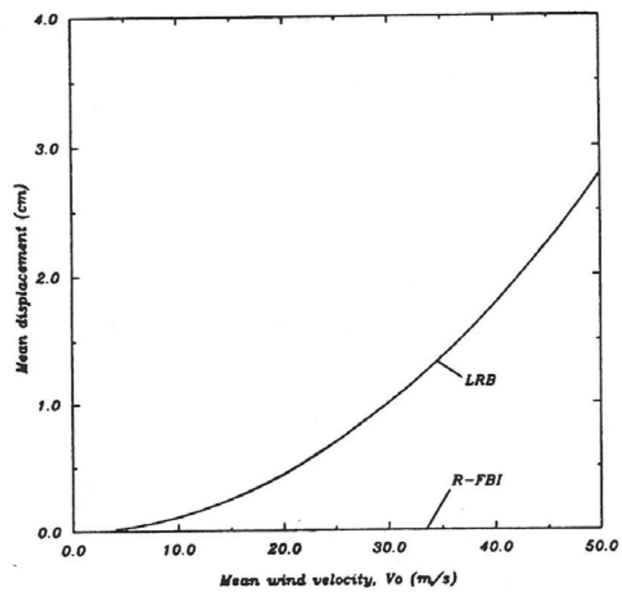

FIG. 5. Mean Displacement Responses for LRB and R-FBI Base-Isolation Systems

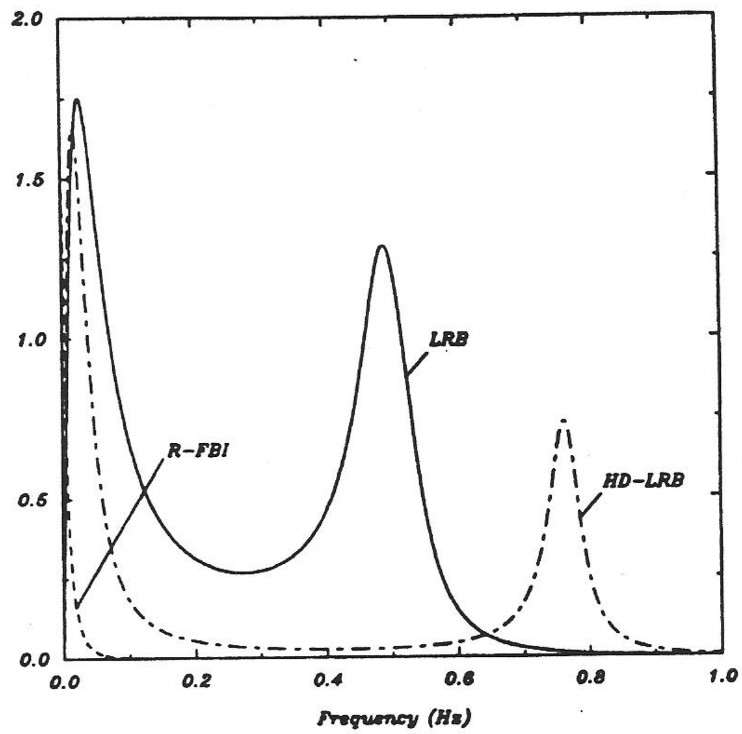

FIG. 6. Displacement Power Spectra for LRB, HD-LRB, and R-FBI Base-Isolation Systems for $V_0 = 42$ m/s

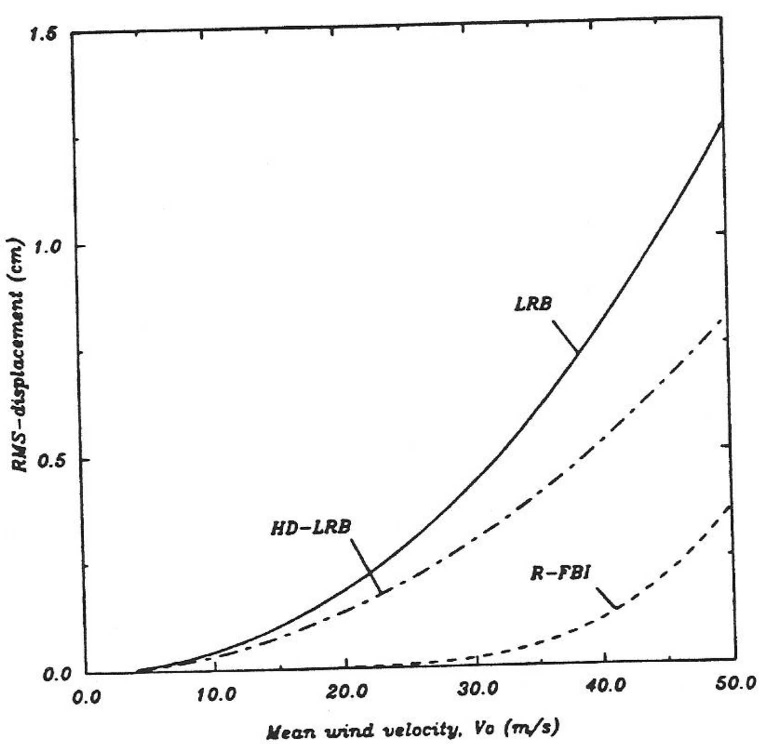

FIG. 9. RMS-Displacements for LRB, HD-LRB, and R-FBI Base-Isolation Systems