Markov Processes

Definition: A stochastic process $X(t)$ is called a Markov process if for every n and for any $t_1 < t_2 < ... < t_n$, its conditional probability satisfies

$$P\{X(t_n) \le x_n \mid X(t_{n-1}), X(t_{n-2}), ..., X(t_1)\} = P\{X(t_n) \le x_n \mid X(t_{n-1})\}.$$

In other words

$$P\{X(t_n) \le x_n \mid X(t) \; for \; all \; t \le t_{n-1}\} = P\{X(t_n) \le x_n \mid X(t_{n-1})\}.$$

Properties:

i) If $X(t)$ is a Markov process, then it is also Markov in reverse. That is, $P\{X(t_1) \le x_1 \mid X(t) \; for \; all \; t \ge t_2\} = P\{X(t_1) \le x_1 \mid X(t_2)\}$.

ii) For a Markov process, the future is independent of the past under the given condition of present.

iii) If for any $t_1 < t_2$, $X(t_2) - X(t_1)$ be independent of $X(t)$ for $t \le t_1$, then $X(t)$ is a Markov process. Thus, independent increment processes (such as Poisson and Wiener-Levy processes) are Markov processes.

iv) If $X(t)$ is a Markov process, then $E\{X(t_n) \mid X(t_{n-1}), ..., X(t_1)\} = E\{X(t_n) \mid X(t_{n-1})\}$.

v) A Markov process $X(t)$ is always associated with a first-order equation

$$\frac{dX}{dt} - \beta(X,t) = j(t), \text{ with } j(t) = \frac{dW}{dt} \text{ or } j(t)dt = dW,$$

where $W(t)$ is an independent increment process (i.e. $j(t_1)$, $j(t_2)$, …, $j(t_n)$ are independent random variables for any t_1, t_2,…, t_n and any n).

The solution to the first-order equation is formally given as

$$X(t) = X(t_0) + \int_{t_0}^{t} \beta(X(\tau),\tau)d\tau + \int_{t_0}^{t} j(\tau)d\tau .$$

Thus $X(t_1)$ is uniquely determined from $X(t_0)$ and $j(t)$ with $t_0 < t \le t_1$. $X(t_0)$ depends on $j(t)$ for $t \le t_0$ and hence is independent of $j(t)$ for $t > t_0$. Thus, given $X(t_0)$ the past of the process $X(t)$ (i.e. $X(t)$ for $t < t_0$) has no effect on the future of $X(t)$ (i.e. for $t > t_0$). That is, $X(t)$ is a Markov process.

vi) A continuous random process $X(t)$ is said to Markovian if its conditional probability density satisfies the relation

$$f_X(x_n, t_n \mid x_{n-1}, t_{n-1}; ...; x_2, t_2; x_1, t_1) = f_X(x_n, t_n \mid x_{n-1}, t_{n-1}) \text{ for any } t_1 < t_2 < ... < t_n .$$

vii) The following relationships hold for a Markov process:

a. $f_{\mathbf{X}}(x_1, t_1; x_2, t_2; x_3, t_3) = f_X(x_3, t_3 \mid x_2, t_2) f_X(x_2, t_2 \mid x_1, t_1) f_X(x_1, t_1),$

b. $f_{\mathbf{X}}(x_1, t_1; ...; x_n, t_n) = f_X(x_n, t_n \mid x_{n-1}, t_{n-1}) ... f_X(x_2, t_2 \mid x_1, t_1) f_X(x_1, t_1),$

where $f_X(x_n, t_n \mid x_{n-1}, t_{n-1})$ is called the transition probability density.

c. If $X(0) = x_o$, then $f_X(x_1, t_1) = f_X(x_1, t_1 \mid x_o, 0)$.

Thus, a Markov process $X(t)$ is fully specified in any of the following three equivalent ways:

i) Given the first-order density and the transition probability density.
ii) Given the second-order density $f_{\mathbf{X}}(x_1, t_1; x_2, t_2)$.
iii) Given the transition probability density and $X(0)$.

The Chapman-Kolmogorov-Smoluchowski Equation

For any continuous random process

$$\begin{aligned} f(x,t \mid x_0, t_0) &= \int_{-\infty}^{+\infty} f(x,t; x_1, t_1 \mid x_0, t_0) dx_1 \\ &= \int_{-\infty}^{+\infty} f(x,t \mid x_1, t_1; x_0, t_0) f(x_1, t_1 \mid x_0, t_0) dx_1 \end{aligned}.$$

For a Markov process

$$f(x,t \mid x_1, t_1; x_0, t_0) = f(x,t \mid x_1, t_1).$$

Thus,

$$f(x,t \mid x_0, t_0) = \int_{-\infty}^{+\infty} f(x,t \mid x_1, t_1) f(x_1, t_1 \mid x_0, t_0) dx_1 .$$

This integral equation for the transition probability density of a Markov process is called the Chapman-Kolmogorov-Smoluchowski equation.

Fokker-Planck Equation

It may be shown that the transition probability density must satisfy the following (forward) Fokker-Planck equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\left[\alpha_1(x,t) f\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\alpha_{11}(x,t) f\right].$$

Here α_1 and α_{11} are the incremental moments defined as

$$\alpha_1(x,t) = \lim_{dt\to 0}\frac{1}{dt}E\{dX(t) \mid X(t) = x\},$$

$$\alpha_{11}(x,t) = \lim_{dt\to 0}\frac{1}{dt}E\left\{[dX(t)]^2 \mid X(t) = x\right\}.$$

The Kolmogorov (Backward) equation is given as

$$\frac{\partial f}{\partial t_0} + \alpha_1(x_0,t_0)\frac{\partial f}{\partial x_0} + \frac{\alpha_{11}(x_0,t_0)}{2}\frac{\partial^2 f}{\partial x_0^2} = 0$$

Note that

$$f = f(x,t \mid x_0,t_0).$$

Example: Determine the Fokker-Planck equation for the Wiener-Levy process and find the transition probability density function.

Recalling that

$$E\{W(t)\} = 0, \quad E\{W^2(t)\} = 2Dt, \qquad R_{WW}(t_1,t_2) = \begin{Bmatrix} 2Dt_1 & t_2 \ge t_1 \\ 2Dt_2 & t_1 \ge t_2 \end{Bmatrix},$$

and

$$E\{W(t_2) - W(t_1)\} = 0, \qquad E\left\{[W(t_2) - W(t_1)]^2\right\} = 2D(t_2 - t_1) \text{ for } t_2 > t_1.$$

Let

$$t_2 = t + dt,\ t_1 = t,\ dW = W(t+dt) - W(t)$$

Thus

$$E\{dW\} = 0,\ E\left\{(dW)^2\right\} = 2Ddt$$

Now

$$\alpha_1 = \lim_{dt \to 0} \frac{1}{dt} E\{dW \mid W\} = 0\,,$$

$$\alpha_{11} = \lim_{dt \to 0} \frac{1}{dt} E\{(dW)^2 \mid W\} = 2D\,.$$

The corresponding Fokker-Planck equation then becomes

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(\alpha_1 f) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\alpha_{11} f) = D\frac{\partial^2 f}{\partial x^2}\,.$$

Using w for Wiener process instead of x, it follows that

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial W^2}\,.$$

For $f(w,t_o \mid w_o,t_o) = \delta(w - w_o)$, the solution becomes

$$f = \frac{e^{-\frac{(w-w_o)^2}{4D(t-t_o)}}}{\sqrt{4\pi D(t-t_o)}}\,.$$

Fokker-Planck Equation for a Vector Markov Process

The transition probability density function of a vector Markov process $\mathbf{X}(t)$ satisfies the following Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[\alpha_j(\mathbf{x},t) f\right] + \frac{1}{2}\sum_i \sum_j \frac{\partial^2}{\partial x_i \partial x_j}\left[\alpha_{ij}(\mathbf{x},t) f\right].$$

The incremental moments α_j and α_{ij} are given as

$$\alpha_j = \lim_{dt \to 0} \frac{1}{dt} E\{dX_j(t) \mid \mathbf{X}(t) = \mathbf{x}\},$$

$$\alpha_{ij} = \lim_{dt \to 0} \frac{1}{dt} E\{dX_i(t) dX_j(t) \mid \mathbf{X}(t) = \mathbf{x}\}.$$

Fokker-Planck Equation for Ito's Equation

Consider Ito's equation

$$\frac{d\mathbf{X}}{\mathbf{dt}} = \mathbf{g}(\mathbf{X},t) + \mathbf{G}(\mathbf{x},t)\cdot\mathbf{n}(t)$$

or

$$\frac{dX_i}{dt} = g_i(\mathbf{X},t) + \sum_j G_{ij}(\mathbf{X},t)n_j(t)$$

or

$$d\mathbf{X} = \mathbf{g}(\mathbf{X},t)dt + \mathbf{G}(\mathbf{X},t)\cdot\mathbf{dW}\ .$$

Here, $\mathbf{n}$ and $\mathbf{W}$ are vector white noise and Wiener process with

$$E\{n_i\} = E\{dW_i\} = 0$$

$$E\{n_i(t+\tau)n_j(t)\} = 2D_{ij}\delta(\tau),\ E\{dW_i dW_j\} = 2D_{ij}dt\ .$$

The incremental moments are then given as

$$a_j = \lim_{dt\to 0}\frac{1}{dt}E\{dX_j \mid \mathbf{X}=\mathbf{x}\} = g_j(x,t),$$

$$a_{ij} = \lim_{dt\to 0}\frac{1}{dt}E\{dX_i dX_j \mid \mathbf{X}=\mathbf{x}\},$$

$$\begin{aligned} dX_i dX_j &= g_i g_j (dt)^2 + g_i\sum_k G_{jk}dW_k dt \\ &\quad + g_j\sum_k G_{ik}dW_k dt + \sum_k\sum_\ell G_{ik}G_{j\ell}dW_k dW_\ell \end{aligned}.$$

Then

$$\alpha_{ij} = 2\sum_k\sum_\ell G_{ik}G_{j\ell}D_{k\ell} = 2\left(\mathbf{G}\cdot\mathbf{D}\cdot\mathbf{G}^T\right)_{ij}$$

The Fokker-Planck equation then becomes

$$\frac{\partial f}{\partial t} = -\sum_j\frac{\partial}{\partial x_j}\left[g_j(\mathbf{x},t)f\right] + \sum_i\sum_j\frac{\partial^2}{\partial x_i\partial x_j}\left[\left(\mathbf{G}\cdot\mathbf{D}\cdot\mathbf{G}^T\right)_{ij} f\right].$$

Example

Consider a first order system

$$\frac{dx}{dt} = g(x,t) + G(x,t)\frac{dW}{dt}. \qquad E\{(dW)^2\} = 2Ddt.$$

The Fokker-Planck equation is given by

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(gf) + D\frac{\partial^2}{\partial x^2}(G^2 f).$$

For stationary solution we find,

$$\frac{d}{dx}\left[-gf + D\frac{d}{dx}(G^2 f)\right] = 0,$$

$$-gf + D\frac{d}{dx}(G^2 f) = c_1 = 0,$$

Let

$$G^2 f = F,$$

$$D\frac{dF}{dx} = \frac{g}{G^2}F,$$

$$\frac{dF}{F} = \frac{g}{DG^2}dx,$$

$$F = C\exp\left\{+\int_0^x \frac{g(x_1)}{DG^2(x_1)}dx_1\right\},$$

and finally

$$f = \frac{c}{G^2}\exp\left\{\int_0^x \frac{g(x_1)}{DG^2(x_1)}dx_1\right\}.$$