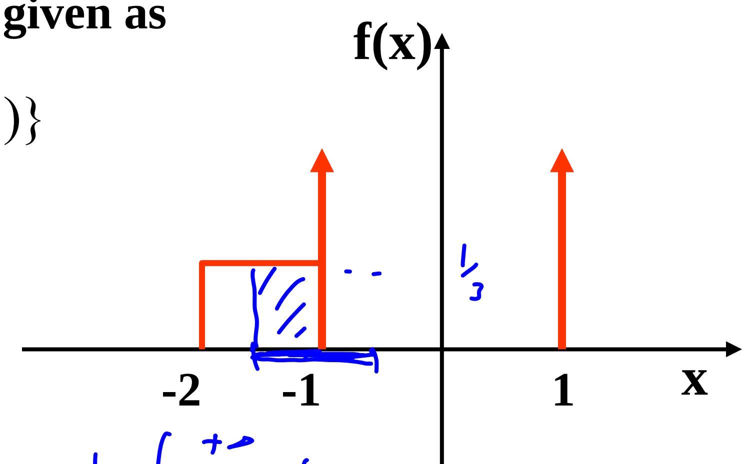
ME 529 **EXAM 1** **October 15, 2001**

The probability density of a random variable X is given as

$$f_x(x) = \frac{1}{3}\{[u(x+2) - u(x+1)] + \delta(x-1) + \delta(x+1)\}$$



a) Evaluate mean and variance of X

b) Evaluate $\varphi_x(\omega)$

c) Evaluate P{-1.5< X <-0.5}.

a)
$$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_{-2}^{-1} x\left(\frac{1}{3}\right)dx + \frac{1}{3}\int_{-\infty}^{+\infty} x\left(\delta(x-1) + \delta(x+1)\right)dx$$

$$= \frac{1}{3}\frac{x^2}{2}\Big|_{-2}^{-1} + \frac{1}{3}(1-1) = \frac{1}{6}(1-4) = -1/2$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_{-2}^{-1} x^2\left(\frac{1}{3}\right)dx + \frac{1}{3}(1+1) = \frac{x^3}{9}\Big|_{-2}^{-1} + 2/3$$

$$= 13/9$$

$$\sigma_x^2 = E(X^2) - (E(X))^2 = \frac{13}{9} - (-1/2)^2 = \frac{43}{36}$$

b)
$$\Phi_x(\omega) = E(e^{i\omega x}) = \int_{-\infty}^{+\infty} e^{i\omega x} f(x)\,dx = \int_{-2}^{-1} \frac{1}{3} e^{i\omega x}dx + \frac{1}{3}\left(e^{i\omega} + e^{-i\omega}\right)$$

$$\phi_X(\omega) = \frac{1}{3}\left.\frac{e^{i\omega x}}{i\omega}\right|_{-2}^{-1} + \frac{1}{3}\left(2\cos\omega\right) = \frac{1}{3}\left[\frac{e^{-i\omega}}{i\omega} - \frac{e^{-2i\omega}}{i\omega}\right] + \frac{2}{3}\cos\omega$$

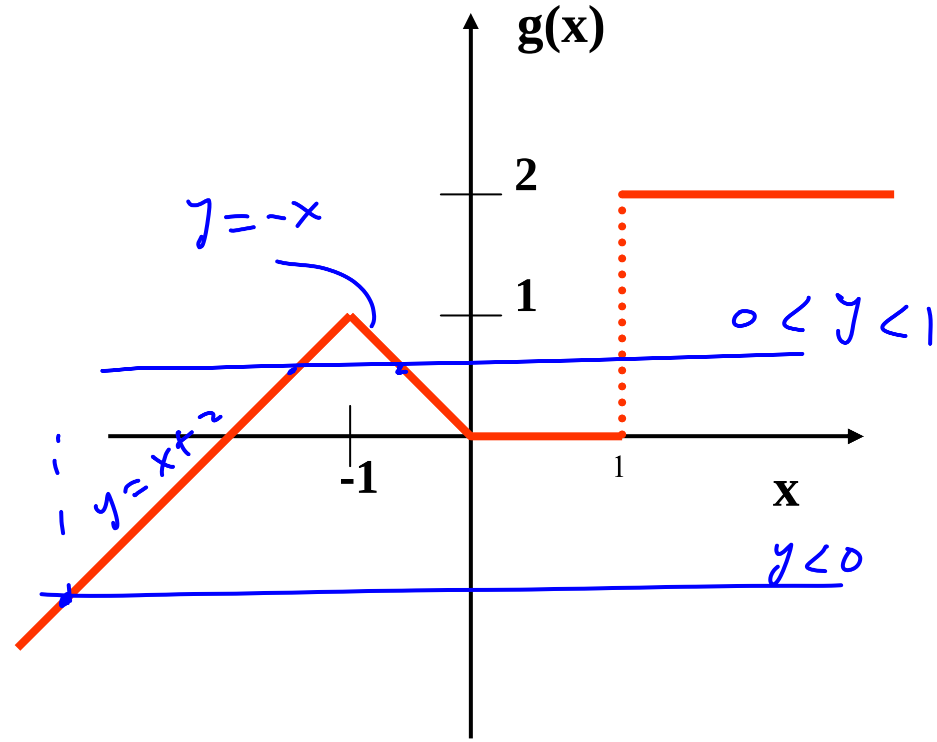
c) $$P\left(-1.5 \le X \le -0.5\right) = \int_{-1.5}^{-0.5} f(x)\,dx$$

$$= \int_{-1.5}^{-1} \frac{1}{3}\,dx + \frac{1}{3} = \frac{1}{3}x\Big|_{-1.5}^{-1} + \frac{1}{3} = -\frac{1}{3} + \frac{1.5}{3} + \frac{1}{3}$$

$$= \frac{1}{2}$$

2) For

$$Y = g(X) = \begin{cases} X+2 & \text{for} \quad X \le -1 \\ -X & \text{for} \quad -1 \le X \le 0 \\ 0 & \text{for} \quad 0 \le X < 1 \\ 2 & \text{for} \quad X \ge 1 \end{cases}$$



a) Determine the probability density of Y in terms of density of X.

b) For $f_x = e^{-x}u(x)$, **determine** $f_y(y)$.

c) Find E{Y}

c) $y<0$ $\qquad y = x+2 \rightarrow$

$x = y-2$, $g' = 1$

$$f_Y(y) = \sum_i \frac{f_X(x_i(y))}{|g'(x_i)|}$$

$$f_Y(y) = \frac{f_X(y-2)}{1} = f_X(y-2)$$

$0 < y < 1$ $\qquad x = y-2$, $x = -y \rightarrow g' = -1$ $\qquad f_Y(y) = f_X(y-2) + f_X(-y)$

$y = 0 \rightarrow 0 \le X \le 1$ $\qquad P(Y=0) = P(0 \le X \le 1) = F_X(1) - F_X(0)$

$f_Y(y) = (F_X(1) - F_X(0))\,\delta(y)$

$y = 2 \rightarrow X > 1$ $\qquad P(Y=2) = P(X>1) = 1 - F_X(1)$

$1 < y < 2$ $\quad f_Y(y) = 0$, $y > 2$ $\quad f_Y(y) = 0$ $\qquad f_Y(y) = (1 - F_X(1))\,\delta(y-2)$

$$f_Y(y) = f_X(y-2)\,U(1-y) + f_X(-y)\,U(y)\,U(1-y)$$

$$+\left(F_X(1) - F_X(0)\right)\delta(y) + \left(1 - F_X(1)\right)\delta(y-2)$$

b) $f_X(x) = e^{-x}U(x)$, $F_X(x) = (1-e^{-x})\,U(x)$, $F_X(0)=0$

$$f_Y(y) = e^{-(y-2)}\underbrace{U(y-2)\,U(1-y)}_{0} + e^{y}\,U(-y)\cancel{U(y)}\,U(1-y)$$ 0

$$+(1-e^{-1})\,\delta(y) + e^{-1}\,\delta(y-2)$$

$$f_Y(y) = (1-e^{-1})\,\delta(y) + e^{-1}\,\delta(y-2)$$

c) $$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\,dx = \int_{-\infty}^{+\infty} y\left((1-e^{-1})\,\delta(y) + e^{-1}\,\delta(y-2)\right)dy$$

$$= (1-e^{-1})(0) + 2e^{-1} = 2e^{-1}$$

$$E(Y) = E(g(X)) = \int_{-\infty}^{+\infty} g(x)\,f_X(x)\,dx$$

Given $f_{xy}(x,y)$ **and** $Z=XY^3$ **,**

a) Find $f_{zy}(z,y)$ **and** $f_z(z)$**.**

b) Evaluate $f_{zy}(z,y)$ **and** $f_z(z)$ **for** $f_{xy}(x,y)=y^3e^{-xy^3}u(x)u(y)u(1-y)$.

c) Evaluate E(Z) and E(X)

d) Find $P\{-2 < X \le \infty \cap -1 < Y \le 0.5)$.

{Hint $\int_0^\infty x^m e^{-ax}dx = \dfrac{m!}{a^{m+1}}$ **}.**

$y = w$

$x = z/w^3$

$z = xy^3, \quad w = y$

$$J = \begin{vmatrix} y^3 & 3xy^2 \\ 0 & 1 \end{vmatrix} = y^3 = w^3$$

a) $$f_{zw}(z,w) = \sum \frac{f_{xy}(x_i(z,w), y_i(z,w))}{|J|}$$

$$f_{zw}(z,w) = \frac{f_{xy}(z/w^3, w)}{w^3}$$

$$f_z(z) = \int_{-\infty}^{+\infty} \frac{1}{w^3} f_{xy}(z/w^3, w)\, dw$$

b) $$f_{xy}(x,y) = y^3 e^{-xy^3} U(x)\, U(y)\, U(1-y)$$

$$f_{zw}(z,w) = \frac{1}{\cancel{w^3}}\left(\cancel{w^3}\, e^{-\frac{z}{w^3}w^3}\, U(z/w^3)\, U(w)\, U(1-w)\right)$$

$$f_{zw}(z,w) = e^{-z}\, U(z)\, U(w)\, U(1-w), \quad f_z(z) = e^{-z}\, U(z)$$

c) $E(Z) = \int_{-\infty}^{\infty} z f_Z(z)\, dz = \int_0^{\infty} z e^{-z}\, dz = \frac{e^{-z}}{-1}\left(z - \frac{1}{-1}\right)\Big|_0^{\infty} = 1$

$E(Y) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, y f_{XY}(x,y) = \int_{-\infty}^{+\infty} dz \int_{-\infty}^{\infty} dy\, y f_{ZY}(z,y)$

$= \underbrace{\int_{-\infty}^{+\infty} dz\, e^{-z} U(z)}_{} \int_{-\infty}^{+\infty} y\, U(y) U(1-y)\, dy = \int_0^1 y\, dy = \frac{1}{2}$

$E(X) = \iint x f_{XY}(x,y)\, dx\, dy = \int_0^1 dy \int_0^{\infty} y^3 x e^{-xy^3} dx = \int_0^1 dy\, y^3 \frac{1}{y^6}$

$\int_0^{\infty} x e^{-ax} dx = \frac{1}{a^2}$

$E(X) = \int_0^1 \frac{1}{y^3} dy = -\frac{1}{2y^2}\Big|_0^1 = \infty$

d) $P(-2 < X < \infty \cap -1 < Y < 0.5) = \int_{-2}^{\infty} dx \int_{-1}^{0.5} dy\, f_{XY}(x,y)$

$= \int_0^{\infty} dx \int_0^{0.5} dy\, y^3 e^{-xy^3} = 0.5$