ME 529 - Stochastics

Conditional Distributions and Densities

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Conditional Distributions

Outline

- Conditional Expected Value
- Chapman – Kolmogorov Equation
- Sample Mean and Variance
- Estimating Mean and Variance of Random Data
- Alternative Definition For Probability Density Function

Conditional Distributions

Conditional Distribution of *Y* given event *m*

$$F_Y(y \mid m) = P\{Y \le y \mid m\} = \frac{P\{Y \le y \cap m\}}{P(m)}$$

For m = {X ≤ x},

$$F_Y(y \mid X \le x) = \frac{P\{X \le x \cap Y \le y\}}{P\{X \le x\}} = \frac{F_{XY}(x,y)}{F_X(x)}$$

$$f_Y(y \mid X \le x) = \frac{\frac{\partial F_{XY}(x,y)}{\partial y}}{F_X(x)} = \frac{1}{F_X(x)} \int_{-\infty}^{x} f_{XY}(x_1, y)\,dx_1$$

Similarly:

$$F_Y(y \mid x_1 < X \le x_2) = \frac{\int_{x_1}^{x_2} f_{XY}(x,y)\,dx}{F_X(x_2) - F_X(x_1)}$$

Conditional Distributions

Conditional Distribution & Density given X = x

Noting $F_Y(y \mid X = x) = \lim_{\Delta x \to 0} F_Y(y \mid x < X \le x + \Delta x)$,

$$F_Y(y \mid x = x) = \lim_{\Delta x \to 0} \frac{F_{XY}(x + \Delta x, y) - F_{XY}(x,y)}{F_X(x + \Delta x) - F_X(x)} = \frac{\frac{\partial F_{XY}(x,y)}{\partial x}}{\frac{dF_X(x)}{dx}}$$

$$F_Y(y \mid x = x) = \frac{\int_{-\infty}^{y} f_{XY}(x, y_1)\,dy_1}{f_X(x)}$$

$$f_Y(y \mid x) = f_Y(y \mid x = x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Similarly:

$$F_X(x \mid y) = \frac{\int_{-\infty}^{x} f_{XY}(x_1, y)\,dx_1}{f_Y(y)}$$

$$f_X(x \mid y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

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Conditional expected value of a function of a random variable g(Y)

$$E\{g(Y)|m\}=\int_{-\infty}^{+\infty}g(y)f_Y(y|m)dy$$

$$E\{g(Y)|X=x\}=\int_{-\infty}^{+\infty}g(y)f_Y(y|x=x)dy$$

Conditional Expected Value given X = x

$$E\{g(Y)|X=x\}=\frac{1}{f_X(x)}\int_{-\infty}^{+\infty}g(y)f_{XY}(x,y)dy$$

Chapman – Kolmogorov Equation

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Noting $f_X(x|y)=\frac{f_{XY}(x,y)}{f_Y(y)}$ **,**

$$f_X(x|y,z)=\frac{f_{XY}(x,y|z)}{f_Y(y|z)}$$

or

$$f_{XY}(x,y|z)=f_X(x|y,z)f_Y(y|z)$$

Integrating over y:

$$f_X(x|z)=\int_{-\infty}^{+\infty}f_X(x|y,z)f_Y(y|z)dy$$

For Markov processes:

$$f_X(x|y,z)=f_X(x|y) \Rightarrow f_X(x|z)=\int_{-\infty}^{+\infty}f_X(x|y)f_Y(y|z)dy$$

Sample Mean & Variance

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$$\overline{X}=\frac{X_1+X_2+\ldots+X_n}{n} \qquad \overline{V}=\frac{\left(X_1-\overline{X}\right)^2+\ldots+\left(X_n-\overline{X}\right)^2}{n}$$

If X_i have the same mean & variance and form a sequence of uncorrelated random variables:

$$E\{\overline{X}\}=\eta \qquad \sigma_{\overline{X}}^2=\frac{\sigma^2}{n} \qquad E\{\overline{V}\}=\frac{n-1}{n}\sigma^2$$

Statistics of Sample Mean & Variance

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When X_i are jointly normal with

$$f(x_1,\ldots x_n)=\frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n}\exp\left\{-\frac{x_1^2+x_2^2+\ldots+x_n^2}{2\sigma^2}\right\}$$ **, density functions**

of $\overline{X}=\frac{1}{n}\sum_{j=1}^{n}X_j$ **and** $\overline{V}=\frac{1}{n}\sum_j\left(X_j-\overline{X}\right)^2$ **become**

$$f_{\overline{X}}(\overline{x})=\frac{1}{\sqrt{\frac{2\pi\sigma^2}{n}}}exp\left\{-\frac{n\overline{x}^2}{2\sigma^2}\right\}$$

$$f_{\overline{V}}(v)=\frac{1}{2^{\frac{(n-1)}{2}}\left(\frac{\sigma}{\sqrt{n}}\right)^{n-1}\Gamma\left(\frac{n-1}{2}\right)}v^{\frac{(n-3)}{2}}e^{-\frac{nv}{2\sigma^2}}U(v)$$

Chi- and Chi – Square Statistics

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$$\chi = \sqrt{\sum_{j=1}^{n} X_j^2}$$

$$\chi^2 = Y = \sum_{j=1}^{n} X_j^2$$

Density functions of χ and $\chi^2 = Y$:

$$f_X(\chi) = \frac{2}{2^{\frac{n}{2}} \sigma^n \Gamma\left(\frac{n}{2}\right)} \chi^{n-1} e^{-\frac{\chi^2}{2\sigma^2}} U(\chi)$$

$$f_{X^2}(y) = \frac{1}{2^{\frac{n}{2}} \sigma^n \Gamma\left(\frac{n}{2}\right)} y^{\frac{(n-2)}{2}} e^{-\frac{y}{2\sigma^2}} U(y)$$

Estimating Mean & Variance of Random Data

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$$\overline{X} = \frac{\sum_i X_i}{n}$$

$$S^2 = \frac{\sum_i \left(X_i - \overline{X}\right)^2}{n-1}$$

If X_i have η and σ as mean and variance:

$$E\{\overline{X}\} = \eta$$

$$\sigma_{\overline{X}}^2 = \frac{\sigma^2}{n}$$

Centeral Limit Theorem

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***Theorem*: If $\overline{X}$ is the mean of a random sample of size n taken from a population having mean η and variance σ^2 , then $Z = \frac{\overline{X} - \eta}{\sigma / \sqrt{n}}$ is a random variable whose distribution approaches a standard normal distribution as n $\to \infty$, i.e.**

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\}$$

$$P\{|Z| \le z^2\} = 2erf(z)$$

Note: $P\{|Z| \le 1\} \approx 0.68$ $P\{|Z| \le 2\} = 0.85$ $P\{|Z| \le 3\} = 0.997$

Sample Size

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Size of a Sample for a Required Accuracy

Let error $E = |\overline{X} - \eta|$ and the set $|Z| \le z$ lead to $E \le z\frac{\sigma}{\sqrt{n}}$. The sample size needed is $n = \frac{z^2\sigma^2}{E^2}$; if this is the sample size, then with probability of *2erf(z)* the error will not be more than *E*.

***Example*: Let z = 3, σ = 2, E = 0.01, then n = (9)(4)/10^{-4} data points are needed to estimate mean with probability 0.997 and error < 0.01**

Alternative Probability Density Function

Stratonovich definition of Probability Density Function

$$f_X(x) = E\{\delta(X-x)\}$$

$$E\{\delta(X-x)\} = \int_{-\infty}^{+\infty} \delta(x_1 - x) f_X(x_1) dx_1 = f_X(x)$$

$$E\{g(x)\} = E\left\{\int_{-\infty}^{+\infty} g(x)\delta(x-X) f_X(x)dx\right\} = \int_{-\infty}^{+\infty} g(x) E\{\delta(x-X)\}dx = \int_{-\infty}^{+\infty} g(x) f_X(x)dx$$

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Alternative Probability Density Function

Transformation of Random Variable Using the New Definition, Y = g(X)

$$f_Y(y) = \int_{-\infty}^{+\infty} \delta[g(x) - y] f_X(x) dx$$

$$\delta[g(x) - y] = \sum_j \frac{\delta(x - x_j)}{|g'(x_j)|}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} \sum_j \delta(x - x_j) f_X(x) dx = \sum_j \frac{f_X(x_j)}{|g'(x_j)|}$$

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Concluding Remarks

- **Conditional Expected Value**
- **Chapman – Kolmogorov Equation**
- **Sample Mean and Variance**
- **Estimating Mean and Variance for Random Data**
- **Alternative Definition For Probability Density Function**

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Thank you!

Questions?

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