ME 529 - Stochastics

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Second-Order Systems (Stationary Solutions)

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2nd Order Systems

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Outline

- Stationary Solution to Fokker-Planck Equation
- Generalized Stationary Solutions
- Additional Exact Solutions
- Non-linear Systems
- Equations with Random coefficients

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Consider a single-degree-of-freedom system with non-linear spring

$$\ddot{X} + \beta \dot{X} + g(X) = n(t)$$

$$R_{nn}(\tau) = 2D\delta(\tau)$$

$$\Rightarrow \begin{cases} \dfrac{dX}{dt} = \dot{X} \\ \dfrac{d\dot{X}}{dt} = -\beta \dot{X} - g(X) + n(t) \end{cases}$$

Fokker-Planck

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(\dot{x} f) + \frac{\partial}{\partial \dot{x}}[(\beta \dot{x} + g(x)) f] + D \frac{\partial^2 f}{\partial \dot{x}^2}$$

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Stationary Density Function satisfies

$$-\dot{x}\frac{\partial f}{\partial x} + \frac{\partial}{\partial \dot{x}}[(\beta \dot{x} + g(x)) f] + D \frac{\partial^2 f}{\partial \dot{x}^2} = 0$$

or

$$\left(-\dot{x}\frac{\partial f}{\partial x} + g(x)\frac{\partial f}{\partial \dot{x}}\right) + \frac{\partial}{\partial \dot{x}}\left[\beta \dot{x} f + D\frac{\partial f}{\partial \dot{x}}\right] = 0$$

$$\Rightarrow \quad \dot{x}\frac{\partial f}{\partial x} + g(x)\frac{\partial f}{\partial \dot{x}} = 0 \qquad \frac{\partial}{\partial \dot{x}}\left(\beta \dot{x} f + D\frac{\partial f}{\partial \dot{x}}\right) = 0$$

$$f = C(x) e^{-\frac{\beta}{2D}\dot{x}^2} \quad \Rightarrow \quad f = C_0 e^{-\frac{\beta}{D}\left(G(x) + \frac{\dot{x}^2}{2}\right)}$$

Additional Exact Stationary Solutions

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Consider

$$\ddot{X} + h(H)\dot{X} + g(x) = n(t)$$

$$H = \frac{\dot{X}^2}{2} + \int_0^x g(\eta)d\eta$$

$$\Rightarrow \quad \begin{cases} \dfrac{dX}{dt} = \dot{X} \\ \dfrac{d\dot{X}}{dt} = -\left(g(X) + h(H)\dot{X}\right) + n(t) \end{cases}$$

Fokker-Planck

$$\frac{\partial f}{\partial t} = -\dot{x}\frac{\partial f}{\partial x} + \frac{\partial}{\partial \dot{x}}\left[\left(g(x) + h(H)\dot{x}\right)f\right] + D\frac{\partial^2 f}{\partial \dot{x}^2}$$

Stationary

$$-\dot{x}\frac{\partial f}{\partial x} + g(x)\frac{\partial f}{\partial \dot{x}} + \frac{\partial}{\partial \dot{x}}\left[h(H)\dot{x}f + D\frac{\partial f}{\partial \dot{x}}\right] = 0$$

Additional Exact Stationary Solutions

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Now set

$$-\dot{x}\frac{\partial f}{\partial x} + g(x)\frac{\partial f}{\partial \dot{x}} = 0 \qquad \frac{\partial}{\partial \dot{x}}\left[h(H)\dot{x}f + D\frac{\partial f}{\partial \dot{x}}\right] = 0$$

Assuming f = f(H)

$$\frac{\partial f}{\partial \dot{x}} = \frac{\partial f}{\partial H}\dot{x} \qquad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial H}g(x)$$

$$\Rightarrow \quad h(H)\dot{x}f + D\frac{\partial f}{\partial H}\dot{x} = 0$$

or

$$h(H)f + D\frac{\partial f}{\partial H} = 0 \qquad \frac{df}{f} = -\frac{1}{D}h(H)dH \qquad f = C_o e^{-\frac{1}{D}\int_0^H h(\xi)d\xi}$$

with

$$C_o = \left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} exp\left\{-\frac{1}{D}\int_0^H h(\xi)d\xi\right\}dxd\dot{x}\right]$$

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Consider

$$\ddot{X}_i + \beta_i h(H)\dot{X}_i + \frac{\partial V(\mathbf{X})}{\partial X_i} = n_i(t)$$

$$R_{n_i n_j}(\tau) = 2D_i\delta_{ij}\delta(\tau) \qquad H = \frac{1}{2}\sum_i \dot{X}_i^2 + V(\mathbf{X}) \qquad \frac{D_i}{\beta_i} = const$$

Solution

$$f = C_0 \exp\left\{-(\beta_i / D_i)\int_0^H f(\xi)d\xi\right\}$$

Consider

$$\ddot{X} + \left(X^2 + 2\dot{X}^2 - \frac{2}{X^2 + 2\dot{X}^2}\right)2D\dot{X} + \frac{2X^3 + X\dot{X}^2}{X^2 + 2\dot{X}^2} = n(t)$$

$$R_{nn}(\tau) = 2D\delta(\tau)$$

Solution

$$f = A\exp\left\{-\left(x^4 + \dot{x}^4 + x^2\dot{x}^2\right)\right\}\left(x^2 + 2\dot{x}^2\right)$$

Additional Exact Stationary Solutions

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Consider

$$\ddot{X} + \left[H_{\dot{x}}h(H) - \frac{H_{\dot{x}\dot{x}}}{H_{\dot{x}}}\right]D\dot{X} + \frac{H_x}{H_{\dot{x}}} = n(t) \qquad H_x = \frac{\partial H}{\partial x} \qquad H_{\dot{x}} = \frac{\partial H}{\partial \dot{x}}$$

F-P

$$\frac{\partial f}{\partial t} = -\dot{x}\frac{\partial f}{\partial x} + \frac{\partial}{\partial \dot{x}}\left\{\left[\left(H_{\dot{x}}h(H) - \frac{H_{\dot{x}\dot{x}}}{H_{\dot{x}}}\right)D\dot{x} + \frac{H_x}{H_{\dot{x}}}\right]f\right\} + D\frac{\partial^2 f}{\partial \dot{x}^2}$$

$$H(x,\dot{x}) > 0 \qquad H_{\dot{x}} > 0$$

Stationary

$$\begin{cases} -\dot{x}\dfrac{\partial f}{\partial x} + \dfrac{\partial}{\partial \dot{x}}\left(\dfrac{H_x}{H_{\dot{x}}}f\right) = 0 \\ \dfrac{\partial}{\partial \dot{x}}\left[\left(H_{\dot{x}}h(H) - \dfrac{H_{\dot{x}\dot{x}}}{H_{\dot{x}}}\right)\dot{x}f + \dfrac{\partial f}{\partial \dot{x}}\right] = 0 \end{cases}$$

Solution

$$f = C_0 \exp\left\{-\int_0^H h(\xi)d\xi\right\}H_{\dot{x}}$$

Additional Exact Stationary Solutions

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Consider the Nonlinear system given as

$$\ddot{X} + \beta \operatorname{sgn} \dot{X} + \left(1 + \frac{\beta}{D}\left|\dot{X}\right|\right) g(X) = n(t)$$

$$\Rightarrow \quad f = C_0 \exp\left\{-\frac{\beta}{D}\left|\dot{X}\right| - \left(\frac{\beta}{D}\right)^{2x} \int_0^x g(\xi) d\xi\right\}$$

Additional Exact Stationary Solutions

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Consider the nonlinear stochastic equation with random coefficient (Yong & Lin, 1987)

$$\ddot{X} + \left[h(\Gamma) + n_1(t)\right]\dot{X} + \omega_o^2\left[1 + n_2(t)\right]X = n_3(t)$$

$$\Gamma = \frac{1}{2}\dot{X}^2 + \frac{1}{2}\omega_o^2 X^2$$

Fokker-Planck Equation

$$-\dot{X}\frac{\partial f}{\partial x} + \frac{\partial}{\partial \dot{X}}\left\{\left[h(\Gamma)\dot{X} - D_{22}\dot{X} + \omega_o^2 X\right] f\right\} + \frac{\partial^2}{\partial \dot{X}^2}\left[\left(\omega_o^4 D_{11} X^2 + D_{22}\dot{X}^2 + D_{33}\right) f\right] = 0$$

Additional Exact Stationary Solutions

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Corresponding Ito's Equation

$$dX = \dot{X} dt$$

$$d\dot{X} = -\left\{\left(h(\Gamma) - D_{22}\right)\dot{X} + \omega_0^2 X\right\} dt + \sqrt{2\left(\omega_0^4 D_{11} X^2 + D_{22}\dot{X}^2 + D_{33}\right)}\, d\hat{W}$$

$$E\left\{\left(d\hat{W}\right)^2\right\} = dt \quad \Rightarrow \quad f(x,\dot{x}) = \frac{C_3}{\sqrt{\frac{\left(2D_{22}\Gamma + D_{33}\right)}{\pi}}} exp\left\{-\int_0^{\Gamma} \frac{h(u) du}{2D_{22}u + D_{33}}\right\}$$

For

$$h(\Gamma) = \beta\Gamma + \alpha \quad \Rightarrow \quad f(x,\dot{x}) = C\left(\frac{2D_{22}\Gamma + D_{33}}{\pi}\right)^{\frac{1}{2}\left(\beta\frac{D_{33}}{2D_{22}^2} - \frac{\alpha}{D_{22}} - 1\right)} exp\left\{-\frac{\beta\Gamma}{2D_{22}}\right\}$$

Or

$$f(x,\dot{x}) = C_o\, exp\left\{-\frac{\beta\Gamma}{2D_{22}}\right\}$$

Additional Exact Stationary Solutions

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For $\ddot{X} + \left(\alpha + \beta X^2\right)\dot{X} + \omega_o^2\left[1 + n_1(t)\right]X = n_2(t)$ $\qquad \frac{D_{22}}{D_{11}} = \frac{\alpha}{\beta}$

Solution $f = C_4\, exp\left\{-\frac{\beta}{2D_{11}}\left(\dot{X}^2 + \omega_o^2 X^2\right)\right\}$

For

$$\ddot{X}_i + h(\Gamma)\dot{X}_i + \omega_i^2 X_i + \sum_{j=1}^{n} n_{ij}(t)\dot{X}_j + \sum_{j=1}^{n} \eta_{ij}(t) X_j = \xi_j(t)$$

$$\Gamma = \frac{1}{2}\sum_{j=1}^{n}\left(\dot{X}_j^2 + \omega_j^2 X_j^2\right)$$

$$\left\langle n_{ij}(t) n_{ij}(t+\tau)\right\rangle = 2D_{11}\delta(\tau) \qquad \left\langle \eta_{ij}(t)\eta_{ij}(t+\tau)\right\rangle = 2D_{22}\delta(\tau) \qquad \left\langle \xi_j(t)\xi_j(t+\tau)\right\rangle = 2D_{33}\delta(\tau)$$

Solution

$$f = C_5\left(2D_{22}\Gamma + D_{33}\right)^{-\frac{1}{2}} \exp\left[-\int_0^{\Gamma} \frac{h(U) dU}{2D_{22}U + D_{33}}\right]$$