A nonlinear system subject to a random excitation is given as

\[ \ddot{X} + 2\beta \dot{X}(t) + \omega_0^2 \left[ X(t) + \alpha \dot{X}(X, \dot{X}) \right] = Y(t) \]

For small \( \varepsilon \), assume a solution in the form of

\[ X(t) = X_0(t) + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + \ldots \]

Substitute in the equation and order as powers of \( \varepsilon \),

\[ \varepsilon^0: \dot{X}_0 + 2\beta \dot{X}_0 + \omega_0^2 X_0 = Y(t) \]

\[ \varepsilon^1: \dot{X}_1 + 2\beta \dot{X}_1 + \omega_0^2 X_1 = -\omega_0^2 \varphi(X_0, \dot{X}_0) \]

\[ \varepsilon^2: \dot{X}_2 + 2\beta \dot{X}_2 + \omega_0^2 X_2 = -\omega_0^2 \left[ \frac{\partial \varphi(X_0, X_1)}{\partial X_0} X_1 + \frac{\partial \varphi(X_0, X_1)}{\partial X_1} \dot{X}_1 \right] \]

Where we used

\[ \varphi(X, \dot{X}) = \varphi(X_0, \dot{X}_0) + \left[ \frac{\partial \varphi(X_0, X_1)}{\partial X_0} X_1 + \frac{\partial \varphi(X_0, X_1)}{\partial X_1} \dot{X}_1 \right] + \varepsilon^2 \ldots \]
**Perturbation Techniques**

The resulting equations are linear and can be solved:

\[ X(0) = X(0) = 0 \]

\[ X_0(t) = X_0(0) = 0 \]

\[ X_0(t) = X_0(0) = 0 \]

\[ X_0(t) = 0 \]

\[ X_0(t) = 0 \]

**Response Statistics**

\[ E\{X(t)\} = E\{X_0(t)\} + \varepsilon E\{X_1(t)\} + \ldots \]

\[ E\{X^2(t)\} = E\{X_0^2(t)\} + 2\varepsilon E\{X_0(t)X_1(t)\} + \ldots \]

\[ R_{XX}(t_1, t_2) = E\{X_0(t_1)X_0(t_2)\} + \varepsilon \left[ E\{X_0(t_1)X_1(t_2)\} + E\{X_0(t_2)X_1(t_1)\} \right] + \varepsilon^2 \ldots \]

**Duffing Oscillator**

**Duffing Oscillator with Gaussian Excitation**

\[ \dot{X} + 2\beta \dot{X} + \alpha_0^2 \left( X + \varepsilon X_1 \right) = Y(t) \]

**Assume**

\[ E\{Y(t)\} = 0 \]

\[ E\{X_0(t)\} = 0 \]

\[ E\{X_1(t)\} = 0 \]

**Stationary Response**

\[ X_0(t) = \int_0^t h(t - \tau)Y(\tau)d\tau = \int_0^t h(t)Y(t - \tau)d\tau \]

\[ X_0(t) = \int_0^t h(t - \tau)X_0(\tau)d\tau = \int_0^t h(t)X_0(t - \tau)d\tau \]

\[ X_0(t) = -\alpha_0^2 \int_0^t h(t - \tau)X_0(\tau)d\tau \]

**For Gaussian Processes**

\[ E\{Y(t)Y(t)\} = R_{YY}(t_1, t_2) \]

\[ R_{XX}(t_1, t_2) = E\{X_0(t_1)X_0(t_2)\} + \varepsilon \left[ E\{X_0(t_1)X_1(t_2)\} + E\{X_0(t_2)X_1(t_1)\} \right] + \varepsilon^2 \ldots \]
This gives the variance of $X$ up to the 1st order in $\varepsilon$. Other statistics can be found similarly.