Vorticity Transport in Turbulent Flows

Vorticity Equations

The Navier-Stokes and continuity equations are given as

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p+\mu\nabla^{2}\mathbf{u} \tag{1}$$

$$\nabla\cdot\mathbf{u}\,. \tag{2}$$

The Navier-Stokes equation may be restated as

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{?}\times\mathbf{u}+\nabla\frac{|\mathbf{u}|^{2}}{2}=-\nabla\frac{p}{\rho}+\nu\nabla^{2}\mathbf{u} \tag{3}$$

where the vector identity

$$\mathbf{u}\cdot\nabla\mathbf{u}=(\nabla\times\mathbf{u})\times\mathbf{u}+\nabla\frac{|\mathbf{u}|^{2}}{2}=\mathbf{?}\times\mathbf{u}+\nabla\frac{|\mathbf{u}|^{2}}{2} \tag{4}$$

is used, and vorticity is defines as

$$\mathbf{?}=\nabla\times\mathbf{u} \tag{5}$$

Taking curl of the Navier-Stokes equation given by (3), we find the vorticity transport equation. That is

$$\frac{\partial \mathbf{?}}{\partial t}+\nabla\times(\mathbf{?}\times\mathbf{u})=\nu\nabla^{2}\mathbf{?} \tag{6}$$

Here the vector identity

$$\nabla\times\nabla\times\mathbf{u}=\nabla\nabla\cdot\mathbf{u}-\nabla^{2}\mathbf{u}=-\nabla^{2}\mathbf{u} \tag{7}$$

is used. Noting that

$$\begin{aligned}\nabla\times(\mathbf{?}\times\mathbf{u})&=\mathbf{u}\cdot\nabla\mathbf{?}-\mathbf{?}\cdot\nabla\mathbf{u}+\mathbf{?}(\nabla\cdot\mathbf{u})-\mathbf{u}(\nabla\cdot\mathbf{?})\\&=\mathbf{u}\cdot\nabla\mathbf{?}-\mathbf{?}\cdot\nabla\mathbf{u}\end{aligned} \tag{8}$$

Equation (6) may be restated as

$$\frac{\partial \mathbf{?}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{?} = \mathbf{?}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{?} \tag{9}$$

or

$$\frac{\partial \mathbf{?}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{?} = \mathbf{d}\cdot\mathbf{?} + \nu\nabla^2\mathbf{?} \tag{10}$$

where **d** is the deformation rate tensor.

During a turbulent motion $\mathbf{u}$ and $\mathbf{?}$ become random functions of space and time, and they may be decomposed into a mean and fluctuating parts. i.e.,

$$\mathbf{u} = \mathbf{U} + \mathbf{u}', \; U_i = \overline{u_i}, \; \overline{u_i'} = 0, \tag{11}$$

$$\mathbf{?} = \mathbf{O} + \mathbf{?}', \; \mathbf{O} = \overline{\mathbf{?}}, \; \overline{\mathbf{?}'} = 0, \tag{12}$$

where $\mathbf{U}$ and $\mathbf{O}$ are the mean quantities and $\mathbf{u}'$ and $\mathbf{?}'$ are the fluctuating parts. As was noted before, a bar on the top of the letter stands for the (time) averaged quantity.

Substituting (11) and (12) into Equation (10) after averaging we find the mean vorticity transport equation. That is

$$\frac{\partial \Omega_i}{\partial t} + U_j \frac{\partial \Omega_i}{\partial x_j} = -\frac{\partial \overline{\omega_i' u_j'}}{\partial x_j} + \frac{\partial \overline{\omega_j' u_i'}}{\partial x_j} + \Omega_j \frac{\partial U_i}{\partial x_j} + \nu \frac{\partial^2 \Omega_i}{\partial x_j \partial x_j} \tag{13}$$

Subtracting (13) from (10) leads to the equation for the vorticity fluctuation field. That is

$$\frac{\partial \omega_i'}{\partial t} + U_j \frac{\partial \omega_i'}{\partial x_j} = -u_j' \frac{\partial \Omega_i}{\partial x_j} - u_j' \frac{\partial \omega_i'}{\partial x_j} + \Omega_j d_{ij}' + \omega_j' D_{ij} + \omega_j' d_{ij}' + \nu \frac{\partial^2 \omega_i'}{\partial x_j \partial x_j} + \frac{\partial \overline{\omega_i' u_j'}}{\partial x_j} - \overline{\omega_j' d_{ij}'} \tag{14}$$

Equation (14) may be used to derive the mean square vorticity or dissipation rate transport equations.

Vorticity Energy Equations

Mean Flow

$$\underbrace{\left(\frac{\partial}{\partial t}+U_j\frac{\partial}{\partial x_j}\right)\left(\frac{1}{2}\Omega_i\Omega_i\right)}_{\text{Convective Transport of mean Vorticity}} = \underbrace{-\frac{\partial}{\partial x_j}\left(\Omega_i\overline{\omega_i' u_j'}\right)}_{\text{transport by turbulence velocity-vorticity interaction}} + \underbrace{\overline{u_j'\omega_i'}\frac{\partial \Omega_i}{\partial x_j}}_{\text{gradient production of fluctuating vorticity}} + \underbrace{\Omega_i\Omega_j D_{ij}}_{\text{stretching of vorticity by mean shear flow}}$$

$$+ \underbrace{\Omega_i\overline{\omega_j' d_{ij}'}}_{\text{stretching of vorticity by mean shear flow}} + \underbrace{\nu\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{2}\Omega_i\Omega_i\right)}_{\text{viscous diffusion}} - \underbrace{\nu\frac{\partial \Omega_i}{\partial x_j}\frac{\partial \Omega_i}{\partial x_j}}_{\text{dissipation of mean vorticity}}$$

where

$-\frac{\partial}{\partial x_j}\left(\Omega_i\overline{\omega_i' u_j'}\right)$ = transport by turbulence velocity-vorticity interaction

$\overline{u_j'\omega_i'}\frac{\partial \Omega_i}{\partial x_j}$ = gradient production of fluctuating vorticity

$\Omega_i\Omega_j D_{ij}$ = stretching of vorticity by mean shear flow

$\Omega_i\overline{\omega_j' d_{ij}'}$ = stretching of turbulence vorticity by turbulence deformation rate

$\nu\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{2}\Omega_i\Omega_i\right)$ = viscous transport (diffusion)

$-\upsilon\frac{\partial \Omega_i}{\partial x_j}\frac{\partial \Omega_i}{\partial x_j}$ = dissipation of mean vorticity

Fluctuating Flow

$$\left(\frac{\partial}{\partial t}+U_j\frac{\partial}{\partial x_j}\right)\left(\frac{1}{2}\overline{\omega'_i\omega'_i}\right)= \underbrace{-\overline{u'_j\omega'_i}\frac{\partial\Omega_i}{\partial x_j}}_{\substack{\text{gradient production}\\ \text{of fluctuating vorticity}}}\underbrace{-\frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{u'_j\omega'_i\omega'_i}\right)}_{\substack{\text{diffusion of turbulence}\\ \text{vorticity by turbulence}}}+\underbrace{\overline{\omega'_i\omega'_j d'_{ij}}}_{\substack{\text{production of turbulence vorticity}\\ \text{by turbulent stretching}}}+\underbrace{\overline{\omega'_i\omega'_j}D_{ij}}_{\substack{\text{production of turbulence vorticity}\\ \text{by mean shear flow stretching}}}$$

$$+\underbrace{\Omega_j\overline{\omega'_i d'_{ij}}}_{\substack{\text{production by mixed}\\ \text{turbulence-mean flow stretching}}}+\underbrace{\nu\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{2}\overline{\omega'_i\omega'_i}\right)}_{\text{diffusion by viscosity}}-\underbrace{\nu\overline{\frac{\partial\omega'_i}{\partial x_j}\frac{\partial\omega'_i}{\partial x_j}}}_{\text{dissipation destruction}}$$

where

$-\overline{u'_j w'_i}\frac{\partial\Omega_i}{\partial x_j}$ = gradient production of fluctuating vorticity

$-\frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{u'_j w'_i w'_i}\right)$ = diffusion of turbulence vorticity by turbulence

$\overline{w'_i w'_j d'_{ij}}$ = production of turbulence vorticity by turbulent stretching

$\overline{w'_i w'_j}D$ = production of turbulence vorticity by mean shear flow stretching

$\Omega_j\overline{w'_i d'_{ij}}$ = production by mixed turbulence-mean flow stretching

$\nu\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{2}\overline{\omega'_i\omega'_i}\right)$ = diffusion by viscosity

$-\nu\overline{\frac{\partial\omega'_i}{\partial x_j}\frac{\partial\omega'_i}{\partial x_j}}$ = dissipation destruction