Turbulent Flow Between Two Parallel Plates

Consider a turbulent flow field between two parallel plates as shown in the figure.

![Diagram of turbulent flow between two parallel plates]

The Reynolds Equation for the mean turbulent motion is given by

$$ U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \frac{\partial u_i u_j}{\partial x_j} $$  \hspace{1cm} (1)

Mean Flow Equations

Let

$$ U = (U(y),0,0) $$  \hspace{1cm} (2)

Equation (1) leads to

x-comp: \quad 0 = -\frac{1}{\rho} \frac{\partial P}{\partial y} - \frac{d}{dy} \frac{u v'}{\nu} + \nu \frac{d^2 U}{dy^2}  \hspace{1cm} (3)

y-comp: \quad 0 = -\frac{1}{\rho} \frac{\partial P}{\partial y} - \frac{d}{dy} \nu \frac{v'}{\nu}  \hspace{1cm} (4)

Integrating Equation (4), we find

$$ \frac{P}{\rho} + \frac{v'^2}{\nu} = P_0, \quad \text{or} \quad \frac{\partial P}{\partial x} = \frac{dP_0}{dx}. $$  \hspace{1cm} (5)

Equation (3) now becomes

$$ 0 = -\frac{1}{\rho} \frac{dP_0}{dx} - \frac{d}{dy} \frac{u v'}{\nu} + \nu \frac{d^2 U}{dy^2}. $$  \hspace{1cm} (6)
Integrating (6) and noting that \( \frac{dU}{dy} \bigg|_{y=0} = \frac{\tau_0}{\rho} = u^+ \), we find

\[
-\frac{y}{\rho} \frac{dP_0}{dx} - u^+ v - \frac{dU}{dy} - u^+ = 0.
\]  

(7)

At the centerline \( y = h \), equation (7) implies

\[
-\frac{h}{\rho} \frac{dP_0}{dx} = u^+.
\]  

(8)

Eliminating \( \frac{dP_0}{dx} \) between (7) and (8), the result is

\[
-\frac{u^+ v}{u^+} + v \frac{dU}{dy} = u^+ \left( 1 - \frac{y}{h} \right)
\]  

(9)

Introducing the dimensionless variable \( \eta = \frac{y}{h} \), equation (9) may be restated as

\[
-\frac{u^+ v}{u^+} + \frac{1}{R^+} \frac{d}{d\eta} (U^+) = 1 - \eta,
\]  

(10)

where

\[
U^+ = \frac{U}{u^+} \text{ and } R^+ = \frac{u^+ h}{\nu}.
\]  

(11)

Alternatively introducing \( y^* = \frac{yu^+}{\nu} \), equation (9) becomes

\[
-\frac{u^+ v}{u^+} + \frac{dU^+}{dy^*} = 1 - \frac{1}{R^*} y^*.
\]  

(12)

For high Reynolds number flows as \( R^* \to \infty \), equations (10) and (12) imply

\[
-\frac{u^+ v}{u^+} = 1 - \eta \quad \text{(as } R^* \to \infty, \eta \sim 1 \text{ (core region)})
\]  

(13)
\[-\frac{u'v'}{u'^2} + \frac{dU^+}{dy^+} = 1 \quad (\text{as } R^* \to \infty, \ y^+ \sim 1 \text{ (surface layer)}) \quad (14)\]

**Law of the Wall**

We expect the solution to (14) to be given as

\[
\text{Law of the Wall:} \quad \begin{cases} U^+ = f(y^+) \\ \frac{-u'v'}{u'^2} = g(y^+) \end{cases}, \quad (15)
\]

with boundary conditions \( f(0) = 0 \) and \( g(0) = 0 \), and the shapes of \( f(y^+) \) and \( g(y^+) \) are to be found experimentally.

**Velocity Defect Law**

In the core region, the turbulent stresses are given by (13) and the mean velocity is given as

\[
\text{Velocity Defect Law:} \quad \frac{U - U_*}{u_*} = F(\eta) \quad (16)
\]

The velocity gradients from (15) and (16) may be found, i.e.

\[
\frac{dU}{dy} = \frac{u'^2}{v} \frac{df}{dy^+} \quad (17)
\]

and

\[
\frac{dU}{dy} = \frac{u^*}{h} \frac{dF}{d\eta}. \quad (18)
\]

**Inertial Sublayer**

From equations (17) and (18) in the limit of \( \eta \ll 1 \) and \( y^+ >> 1 \), we find

\[
\frac{dU}{dy} = \frac{u'^2}{v} \frac{df}{dy^+} = \frac{u^*}{h} \frac{dF}{d\eta} \quad \text{(as } \eta \to 0, \ y^+ \to \infty) \quad (19)
\]

Multiplying (19) by \( \frac{y}{u^*} \), the result is
\[
\frac{y^+}{dy^+} = \eta \frac{dF(\eta)}{d\eta} = \frac{1}{\kappa} = \text{const.} \tag{20}
\]

Solving, we find
\[
F(\eta) = \frac{1}{\kappa} \ln \eta + c_1 \quad \text{for } \eta << 1 \tag{21}
\]

and
\[
f(y^+) = \frac{1}{\kappa} \ln y^+ + c_2 \quad \text{for } y^+ >> 1 \tag{22}
\]

In the inertial sublayer from (13) or (14), we conclude that
\[
\frac{-u' v'}{u'^2} = 1. \tag{23}
\]

**Logarithmic Friction Law**

The velocity defect law and the law of the wall in the inertial sublayer are given as
\[
\frac{U - U_0}{u^*} = \frac{1}{\kappa} \ln \eta + c_1, \tag{24}
\]
\[
\frac{U}{u^*} = \frac{1}{\kappa} \ln y^* + c_2. \tag{25}
\]

Subtracting, we find
\[
\frac{U_0}{u^*} = \frac{1}{\kappa} \ln R^* + c_2 - c_1 \quad (R^* = \frac{u^* h}{v}) \tag{26}
\]

with \(c_1\) and \(c_2\) known, equation (26) is the statement of the logarithmic friction law.

**Pipe Flow**

The law of the wall and the velocity defect law are also valid for turbulent pipe flows. Equations (9) - (26) can be written for pipe flows with the following minor changes:
\[ \eta = \frac{y}{r_0} \text{ and } R^* = \frac{u^* r_0}{\nu}. \]  

Here, \( r_0 \) is the radius of the pipe and \( y \) is the distance from the wall. For pipe flows, \( \kappa = 0.4 \) and equations (24) - (26) become

\[ U^+ = \frac{U}{U_*} = 2.5 \ln y^* + 5, \text{ is valid for up to } \eta = 0.25 \]  

(28)

\[ \frac{U - U_0}{u^*} = 2.5 \ln \eta - 1 \]  

(29)

\[ \frac{U_0}{u^*} = 2.5 R^* + 6 \]  

(30)

**Wake Function**

The wake function is defined as

\[ W(\eta) = 1 - 2.5 \ln \eta + F(\eta). \]  

(31)

where \( F(\eta) \) is the velocity defect law. Experiment shows that

\[ W(\eta) = \frac{1}{2} \left[ \sin \pi \left( \eta - \frac{1}{2} \right) + 1 \right]. \]  

(32)

**Viscous Sublayer**

In the viscous sublayer, the Reynolds stress is negligible. Equation (14) then becomes

\[ \frac{dU^+}{dy^*} = 1. \]  

(33)

or

\[ U^+ = y^+ \]  

(34)
Kolmogorov Scale

In the inertial sublayer $-\overline{u'v'} \approx u'^2$ and $\frac{\partial U}{\partial y} \approx \frac{u^*}{\kappa y}$. The turbulent production then is given as

$$\text{Production} = -u'v' \frac{\partial U}{\partial y} = \frac{u'^3}{\kappa y}. \quad (35)$$

Experiment shows that in the inertial sublayer, production is equal to dissipation:

$$\varepsilon = \frac{u'^3}{\kappa y} \quad (36)$$

Kolmogorov scale $\eta$ is given by

$$\eta = \left( \frac{\nu^3}{\varepsilon} \right)^{\frac{1}{4}} \quad (37)$$

Let

$$\eta^+ = \frac{\eta u^*}{\nu} \quad (37)$$

then

$$\eta^+ = \left( \frac{u'^4 \nu^3}{\nu^4 u^* \kappa y} \right)^{\frac{1}{4}} \quad \text{or} \quad \eta^+ = (\kappa y^+) \frac{1}{4} \quad (38)$$

The turbulent macroscale near the wall is given as

$$\Lambda = \kappa y \quad (39)$$

or

$$\Lambda^+ = \frac{\Lambda u^*}{\nu} = \kappa y^+ \quad (40)$$
Table of variation of the scales near a wall

<table>
<thead>
<tr>
<th>$y^+$</th>
<th>$\eta^+ = (\kappa y^+)^{\frac{1}{4}}$</th>
<th>$\Lambda^+ = \kappa y^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.2</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>1.5</td>
<td>4</td>
</tr>
<tr>
<td>40</td>
<td>2</td>
<td>16</td>
</tr>
<tr>
<td>200</td>
<td>3</td>
<td>80</td>
</tr>
<tr>
<td>1000</td>
<td>4.5</td>
<td>400</td>
</tr>
</tbody>
</table>

From the table and the schematics diagram, it is observed that for $y^+ \leq \frac{1}{\kappa} = 2.5$, $\Lambda^+ < \eta^+$ and a turbulent flow cannot exist.