Turbulent Wake Flow

Equation of motion

$$U_0 \frac{\partial U}{\partial x} + \frac{\partial}{\partial y}\overline{u'v'} = 0 \tag{1}$$

Momentum integral

$$\int_{-\infty}^{+\infty} dy \left(U_0 \frac{\partial}{\partial x}(U - U_0) + \frac{\partial}{\partial y}\overline{u'v'} \right) = 0 \tag{2}$$

or

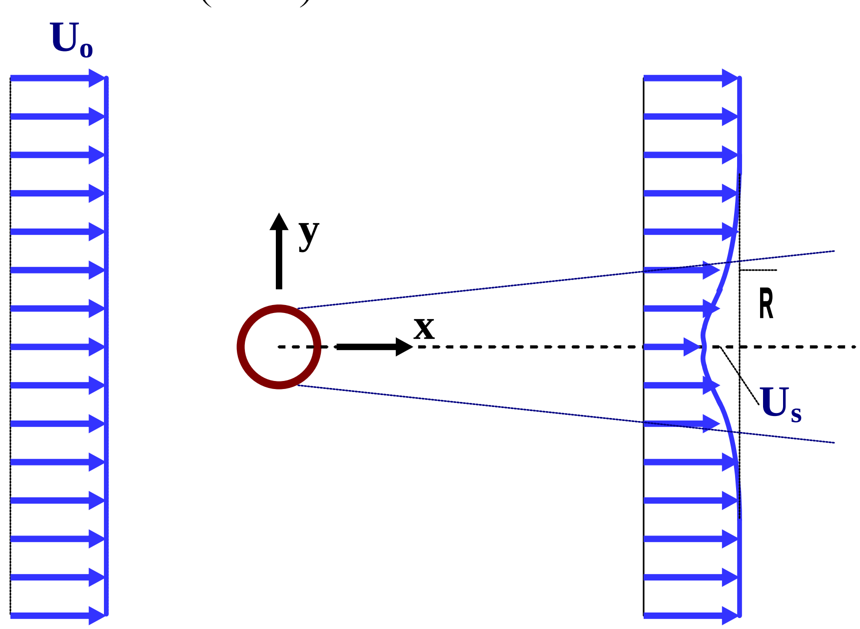
$$\frac{d}{dx}\int_{-\infty}^{+\infty} U_0 (U - U_0) dy = 0 . \tag{3}$$

Integrating Equation (3), it follows that

$$\rho \int_{-\infty}^{+\infty} U_0 (U - U_0) dy = M = -\rho U_0^2 \theta \tag{4}$$

where the momentum thickness is defined as

$$q = \int_{-\infty}^{+\infty} \left(1 - \frac{U}{U_0} \right) dy . \tag{4}$$



The drag coefficient on the cylinder is defined as

$$C_d = \frac{D}{\frac{1}{2}\rho U_0^2 d} = -\frac{M}{\frac{1}{2}\rho U_0^2 d} = \frac{\rho U_0^2 \theta}{\frac{1}{2}\rho U_0^2 d} = \frac{2\theta}{d} \tag{5}$$

For $\mathrm{Re}_d \sim 10^3$ to 3×10^5, $c_d = 1$, and hence

$$q \approx \frac{d}{2}. \tag{6}$$

We consider self-similar solutions given by

$$\frac{U_0 - U}{U_s} = f\left(\frac{y}{\ell}\right), \qquad -\overline{u'v'} = U_s^2 g\left(\frac{y}{\ell}\right) \tag{7}$$

Assuming

$$U_s = Ax^m, \qquad \ell = Bx^n, \tag{8}$$

then

$$\frac{\partial U}{\partial x} \sim \frac{U_s}{x} \sim x^{m-1}, \qquad \frac{\partial \overline{u'v'}}{\partial y} \sim \frac{U_s^2}{\ell} \sim x^{2m-n}. \tag{9}$$

Equations (1) and (4) implies that

$$\begin{matrix} m-1 = 2m-n \\ m+n=0 \end{matrix} \quad \text{or} \quad \begin{Bmatrix} n-m=1 \\ m=-n \end{Bmatrix} \qquad \begin{matrix} n = \frac{1}{2} \\ m = -\frac{1}{2} \end{matrix} \tag{10}$$

Hence,

$$U_0 = Ax^{-\frac{1}{2}}, \qquad \ell = Bx^{\frac{1}{2}} \tag{11}$$

Let

$$\xi = \frac{y}{\ell} = \frac{y}{Bx^{\frac{1}{2}}}, \qquad \frac{\partial \xi}{\partial y} = \frac{1}{Bx^{\frac{1}{2}}}, \quad \frac{\partial \xi}{\partial x} = -\frac{1}{2}\frac{y}{Bx^{-\frac{3}{2}}} = -\frac{\xi}{2x} \tag{12}$$

Then

$$U_0 - U = U_s f(x) = Ax^{\frac{1}{2}} f(x), \tag{13}$$

and

$$-\frac{\partial U}{\partial x} = -\frac{1}{2} Ax^{-\frac{3}{2}} f + Ax^{-\frac{1}{2}} f' \cdot \left(-\frac{\xi}{2x}\right),$$
$$-\frac{\partial U}{\partial x} = -\frac{1}{2} Ax^{-\frac{3}{2}} (f + \xi f') \tag{14}$$

Similarly

$$\overline{u'v'} = -U_s^2 g(\xi) = -A^2 x^{-1} g(\xi), \tag{15}$$

and

$$\frac{\partial \overline{u'v'}}{\partial y} = -\frac{A^2}{x} g' \cdot \left(\frac{1}{Bx^{\frac{1}{2}}}\right) = -\frac{A^2}{Bx^{\frac{3}{2}}} g' \tag{16}$$

Using (14) and (16) in (1), we find

$$\frac{U_0 A}{2} x^{-\frac{3}{2}} (f + \xi f') - \frac{A^2}{Bx^{\frac{3}{2}}} g' = 0 \tag{17}$$

or

$$\frac{U_0 B}{2A} (f + \xi f') = g' \tag{18}$$

To relate g to f, we define an eddy viscosity. i.e.,

$$-\overline{u'v'} = \nu_T \frac{\partial U}{\partial y} \quad \text{or} \qquad \nu_T = -\frac{\overline{u'v'}}{\frac{\partial U}{\partial y}} = -\frac{-U_s^2 g}{-U_s f' \frac{1}{\ell}} = -U_s \ell \frac{g}{f'} \tag{19}$$

That is,

$$\frac{\nu_T}{U_s \ell} = -\frac{g}{f'} = \frac{1}{R_T} \tag{20}$$

where Turbulent Reynolds Number is defines as

$$R_T = \frac{U_s \ell}{\nu_T} \tag{21}$$

Note that in this case

$$\nu_T \approx \text{Cons}\tan t \tag{22}$$

From Equation (20) then

$$g = -\frac{1}{R_T} f'. \tag{23}$$

Using (23) in (18), we find

$$\alpha(f + \xi f') = -f'', \tag{24}$$

where

$$\alpha = \frac{1}{2}\frac{R_T U_0 B}{A} \tag{25}$$

We assume that the constants are selected in such a way that

$$a = 1 \tag{26}$$

Thus, Equation (24) may be restates as

$$(\xi f)' + f'' = 0 \tag{27}$$

Integrating (27), it follows that

$$\xi f + f' = \text{const} = 0 \tag{28}$$

The solution to (28) is given as

$$\frac{U_0 - U}{U_s} = f = e^{-\frac{\xi^2}{2}} \tag{29}$$

The turbulent shear stress as given by (23) then becomes

$$g = -\frac{\xi}{R_T} e^{-\frac{\xi^2}{2}} = -\frac{\overline{u'v'}}{U_s^2} \tag{30}$$

The constant A and B can be evaluated with the use of Equations (4), (25), and (26). That is

$$\theta = \int_{-\infty}^{+\infty} \left(1 - \frac{U}{U_0}\right) dy = \frac{U_s}{U_0} \ell \sqrt{2\pi}\,, \qquad \alpha = 1 = \frac{1}{2} R_T U_0 \frac{B}{A} \tag{31}$$

Thus

$$AB = \frac{\theta U_o}{\sqrt{2\pi}}\,, \qquad \frac{B}{A} = \frac{2}{R_T U_o} \tag{32}$$

or

$$A^2 = \frac{\theta R_T U_o^2}{2\sqrt{2\pi}} \qquad B^2 = \frac{2\theta}{\sqrt{2\pi} R_T} \tag{33}$$

Equation (11) then implies that

$$\left\{ \begin{array}{l} \dfrac{U_s}{U_0} = \left(\dfrac{\theta}{x}\right)^{\frac{1}{2}} \left(\dfrac{R_T}{2\sqrt{2\pi}}\right)^{\frac{1}{2}} = 1.58 \left(\dfrac{\theta}{x}\right)^{\frac{1}{2}} \\ \dfrac{\ell}{\theta} = \left(\dfrac{x}{\theta}\right)^{\frac{1}{2}} \left(\dfrac{2}{\sqrt{2\pi} R_T}\right)^{\frac{1}{2}} = 0.252 \left(\dfrac{x}{\theta}\right)^{\frac{1}{2}} \end{array} \right\}, \qquad R_T \approx 12.5 \text{ (Experiment)} \tag{34}$$

and

$$U_s \ell = \frac{\theta U_0}{\sqrt{2\pi}} = 0.4 \theta U_0\,, \qquad \nu_T = \frac{U_s \ell}{12.5} = 0.0319\, U_0 \theta \tag{35}$$