Chapter 7 • Flow Past Immersed Bodies

7.1 For flow at 20 m/s past a thin flat plate, estimate the distances $x$ from the leading edge at which the boundary layer thickness will be either 1 mm or 10 cm, for (a) air; and (b) water at 20°C and 1 atm.

**Solution:**  (a) For air, take $\rho = 1.2 \text{ kg/m}^3$ and $\mu = 1.8\times10^{-5} \text{ kg/m} \cdot \text{s}$. Guess laminar flow:

$$\frac{\delta_{laminar}}{x} = \frac{5.0}{Re_x^{1/2}}, \quad \text{or:} \quad x = \frac{\delta^2 \rho U}{25\mu} = \frac{(0.001)^2(1.2)(20)}{25(1.8\times10^{-5})} = 0.0533 \text{ m} \quad \text{Ans. (air—1 mm)}$$

Check $Re_x = 1.2(20)(0.0533)/1.8\times10^{-5} = 71,000 \quad \text{OK, laminar flow}$

(a) For the thicker boundary layer, guess turbulent flow:

$$\frac{\delta_{turb}}{x} = \frac{0.16}{(\rho U x/\mu)^{1/7}}, \quad \text{solve for} \quad x = 6.06 \text{ m} \quad \text{Ans. (a—10 cm)}$$

Check $Re_x = 8.1\times10^6, \quad \text{OK, turbulent flow}$

(b) For water, take $\rho = 998 \text{ kg/m}^3$ and $\mu = 0.001 \text{ kg/m} \cdot \text{s}$. Both cases are probably turbulent:

$\delta = 1 \text{ mm:} \quad x_{turb} = 0.0442 \text{ m}, \quad Re_x = 882,000 \quad \text{(barely turbulent)} \quad \text{Ans. (water—1 mm)}$

$\delta = 10 \text{ cm:} \quad x_{turb} = 9.5 \text{ m}, \quad Re_x = 1.9\times10^8 \quad \text{(OK, turbulent)} \quad \text{Ans. (water—10 cm)}$

7.2 Air, equivalent to a Standard Altitude of 4000 m, flows at 450 mi/h past a wing which has a thickness of 18 cm, a chord length of 1.5 m, and a wingspan of 12 m. What is the appropriate value of the Reynolds number for correlating the lift and drag of this wing? Explain your selection.

**Solution:** Convert 450 mi/h = 201 m/s, at 4000 m, $\rho = 0.819 \text{ kg/m} \cdot \text{s}, \quad T = 262 \text{ K,} \quad \mu = 1.66\times10^{-5} \text{ kg/m} \cdot \text{s}$. The appropriate length is the *chord*, $C = 1.5 \text{ m}$, and the best parameter to correlate with lift and drag is $Re_C = (0.819)(201)(1.5)/1.66\times10^{-5} = 1.5\times10^7 \quad \text{Ans.}$

7.3 Equation (7.1b) assumes that the boundary layer on the plate is turbulent from the leading edge onward. Devise a scheme for determining the boundary-layer thickness more accurately when the flow is laminar up to a point $Re_{x, crit}$ and turbulent thereafter. Apply this scheme to computation of the boundary-layer thickness at $x = 1.5 \text{ m}$ in 40 m/s
7.8  Air, \( \rho = 1.2 \text{ kg/m}^3 \) and \( \mu = 1.8 \times 10^{-5} \text{ kg/m s} \), flows at 10 m/s past a flat plate. At the trailing edge of the plate, the following velocity profile data are measured:

<table>
<thead>
<tr>
<th>( y ), mm</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
<th>5.0</th>
<th>6.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u ), m/s</td>
<td>0</td>
<td>1.75</td>
<td>3.47</td>
<td>6.58</td>
<td>8.70</td>
<td>9.68</td>
<td>10.0</td>
<td>10.0</td>
</tr>
<tr>
<td>( u(U-u) ), m²/s</td>
<td>0</td>
<td>14.44</td>
<td>22.66</td>
<td>22.50</td>
<td>11.31</td>
<td>3.10</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

If the upper surface has an area of 0.6 m², estimate, using momentum concepts, the friction drag, in newtons, on the upper surface.

**Solution:** Make a numerical estimate of drag from Eq. (7.2): \( F = \rho b \int u(U-u)dy \). We have added the numerical values of \( u(U-u) \) to the data above. Using the trapezoidal rule between each pair of points in this table yields

\[
\delta \approx \frac{1}{1000} \delta \left[ \frac{1}{2} \left( 0 + 14.44 \right) + \frac{1}{2} \left( 14.44 + 22.66 \right) + \cdots \right] = 0.061 \text{ m}^3 \text{s}^{-1}
\]

The drag is approximately \( F = 1.2b(0.061) = 0.073b \) newtons or **0.073 N/m**. Ans.

7.9  Repeat the flat-plate momentum analysis of Sec. 7.2 by replacing the parabolic profile, Eq. (7.6), with the more accurate sinusoidal profile:

\[
\frac{u}{U} \approx \sin \left( \frac{\pi y}{2 \delta} \right)
\]

Compute momentum-integral estimates of \( C_f \), \( \delta/x \), \( \delta*/x \), and \( H \).

**Solution:** Carry out the same integrations as Section 7.2, but results are more accurate:

\[
\theta = \frac{1}{2 \pi} \int_{0}^{\delta} u \left( \frac{1}{U} - \frac{u}{U} \right)dy = \frac{4 - \pi}{2 \pi} \delta = 0.1366\delta; \quad \delta* = \frac{1}{\pi} \int_{0}^{\delta} \left( 1 - \frac{u}{U} \right)dy = \frac{\pi - 2}{\pi} \delta = 0.3634\delta
\]

\[
\tau_w = \mu \pi U^2 = \rho U^2 \frac{d}{dx} \left[ \frac{4 - \pi}{2 \pi} \delta \right], \quad \text{integrate to:} \quad \frac{\delta}{x} \approx \frac{\pi \sqrt{2}/(4 - \pi)}{\sqrt{Re_x}} \approx \frac{4.80}{\sqrt{Re_x}} \quad \text{(5% low)}
\]

Substitute these results back to obtain the desired (accurate) dimensionless expressions:

\[
\frac{\delta}{x} \approx \frac{4.80}{\sqrt{Re_x}}; \quad C_f = \frac{\theta}{x} \approx \frac{0.655}{\sqrt{Re_x}}; \quad \frac{\delta*}{x} \approx \frac{1.743}{\sqrt{Re_x}}; \quad H = \frac{\delta*}{\theta} = 2.66 \quad \text{Ans. (a, b, c, d)}
\]
7.10 Repeat Prob. 7.9, using the polynomial profile suggested by K. Pohlhausen in 1921:

\[
\frac{u}{U} \approx 2 \frac{y}{\delta} - 2 \frac{y^3}{\delta^3} + \frac{y^4}{\delta^4}
\]

Does this profile satisfy the boundary conditions of laminar flat-plate flow?

**Solution:** Pohlhausen’s quadratic profile satisfies no-slip at the wall, a smooth merge with \( u \to U \) as \( y \to \delta \), and, further, the boundary-layer curvature condition at the wall. From Eq. (7.19b),

\[
\left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2} \right)_{\text{wall}} = 0, \quad \text{or:} \quad \frac{\partial^2 u}{\partial y^2} \bigg|_{\text{wall}} = 0 \quad \text{for flat-plate flow} \quad \left( \frac{\partial p}{\partial x} = 0 \right)
\]

This profile gives the following integral approximations:

\[
\frac{\delta}{x} \approx \frac{\sqrt{1260/37}}{\sqrt{\text{Re}_x}} \approx 5.83 \quad \frac{\delta^k}{x} \approx \frac{5.83}{\sqrt{\text{Re}_x}}
\]

\[
\frac{\partial^2 u}{\partial y^2} \approx \frac{1.751}{\sqrt{\text{Re}_x}} \quad \text{H} \approx 2.554 \quad \text{Ans. (a, b, c, d)}
\]

7.11 Air at 20°C and 1 atm flows at 2 m/s past a sharp flat plate. Assuming that the Kármán parabolic-profile analysis, Eqs. (7.6–7.10), is accurate, estimate (a) the local velocity \( u \); and (b) the local shear stress \( \tau \) at the position \((x, y) = (50 \text{ cm}, 5 \text{ mm})\).

**Solution:** For air, take \( \rho = 1.2 \text{ kg/m}^3 \) and \( \mu = 1.8 \times 10^{-5} \text{ kg/m} \cdot \text{s} \). First compute \( \text{Re}_x \) and \( \delta(x) \): The location we want is \( y/\delta = 5 \text{ mm}/10.65 \text{ mm} = 0.47 \), and Eq. (7.6) predicts local velocity:

\[
u(0.5 \text{ m}, 5 \text{ mm}) = U \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) = (2 \text{ m/s}) \left[ 2(0.47) - (0.47)^2 \right] = 1.44 \text{ m/s} \quad \text{Ans. (a)}
\]

The local shear stress at this \( y \) position is estimated by differentiating Eq. (7.6):

\[
\tau(0.5 \text{ m}, 5 \text{ mm}) = \mu \left( \frac{\partial u}{\partial y} \right) = \frac{\mu U}{\delta} \left( 2 - \frac{2y}{\delta} \right) = \frac{(1.8 \times 10^{-5} \text{ kg/m} \cdot \text{s})(2 \text{ m/s})}{0.01065 \text{ m}} \left[ 2 - 2(0.47) \right] = 0.0036 \text{ Pa} \quad \text{Ans. (b)}
\]
Problem 4:

Hand-wavy way: (ok by me)

Concerned with flow along cylinder in "z" direction

Want to know $V_z$, $\frac{\partial}{\partial \theta}$'s are 0 because axisymmetric

$V_z$ - momentum equation from Appendix D:

$$\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + V \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_z}{\partial r} \right) + \frac{\partial^2 V_z}{\partial z^2} \right)$$

Steady, axisymmetric

$V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial^2 V_z}{\partial z^2} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + V \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_z}{\partial r} \right) + \frac{\partial^2 V_z}{\partial z^2} \right)$

derivatives of $V_z$ with $r$ will probably be much bigger than derivatives with $z$
\[
V_r \frac{\partial v_z}{\partial r} + V_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + V \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) \right)
\]

Maybe \( \frac{\partial p}{\partial z} \) is small because \( \frac{\partial p}{\partial z} \) far away is 0?

Best Guess:

\[
V_r \frac{\partial v_z}{\partial r} + V_z \frac{\partial v_z}{\partial z} = V \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) \right)
\]
Formal Way:

\[ r = R + \delta \hat{r} \]
\[ z = \delta \hat{z} \]
\[ V_z = V_\infty \hat{V}_z \]
\[ V_r = V_\delta \hat{V}_r \]
\[ \frac{\partial}{\partial \hat{r}} = \frac{\partial}{\partial r} \frac{\partial \hat{r}}{\partial r} = \frac{1}{\delta} \frac{\partial}{\partial \hat{r}} \]
\[ \frac{\partial}{\partial \hat{z}} = \frac{\partial^2}{\partial \hat{z}^2} \]

Continuity:

\[ \frac{1}{R + \delta \hat{r}} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left( (R + \delta \hat{r}) V_\delta \hat{V}_r \right) + \frac{1}{\delta} \frac{\partial}{\partial \hat{z}} \left( V_\infty \hat{V}_z \right) = 0 \]

If \( R \gg \delta \)

\[ \frac{1}{R} \frac{1}{\delta} \frac{\partial}{\partial \hat{r}} \left( R V_\delta \hat{V}_r \right) + \frac{1}{\delta} \frac{\partial}{\partial \hat{z}} \left( V_\infty \hat{V}_z \right) = 0 \]

\[ \frac{V_\delta}{\delta} \frac{\partial}{\partial \delta} + \frac{V_\infty}{\delta} \frac{\partial}{\partial \hat{z}} = 0 \]

\[ V_\delta \sim O(\frac{\delta}{\hat{r}} V_\infty) \]

If \( R \sim O(\delta) \)

\( O() \) means about the same size (order of)

Rescale \( R \): \( R = \hat{R} \delta \) with \( \hat{R} = R / \delta \)
\[
\frac{1}{\delta (R^2 + \ell^2)} \frac{1}{\delta} \frac{\partial}{\partial \delta} \left( \delta (R^2 + \ell^2) V_0 \hat{V}_r \right) + \frac{1}{\ell} \frac{\partial}{\partial \ell} \left( V_0 \hat{V}_\ell \right) = 0
\]
\[
\frac{V_0}{\delta} \left[ \frac{1}{R^2 + \ell^2} \frac{\partial}{\partial r} \left( (R^2 + \ell^2) V_r \right) + \frac{V_0}{\ell} \frac{\partial}{\partial \ell} (V_\ell) \right] = 0
\]
\[
\text{All size of } O(\ell)
\]
\[
V_0 = \left( \frac{\delta}{\ell} V_0 \right) \left\{ \text{Same result.} \right. \]

Now r-momentum
\[
V_r \frac{\partial V_r}{\partial r} + V_\ell \frac{\partial V_r}{\partial \ell} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + V \left( \frac{1}{R^2 + \ell^2} \frac{\partial}{\partial r} \left( (R^2 + \ell^2) V_r \right) + \frac{\partial V_r}{\partial \ell} \right) = \frac{V_r}{r^2}
\]

Let \( P = P^* \delta \)
\( P^* \) magnitude of pressure change in \( \ell \)

\[
\frac{V_0^2}{\delta} \left( \frac{\partial V_r}{\partial \ell} + V_\ell \frac{\partial V_r}{\partial \ell} \right) = -\frac{1}{\rho} \frac{\partial P^*}{\partial \ell} + V \left( \frac{1}{R^2 + \ell^2} \frac{\partial}{\partial r} \left( (R^2 + \ell^2) V_r \right) \right.
\]
\[
+ \frac{V_0}{\ell} \frac{\partial V_\ell}{\partial \ell} - \frac{V_\ell}{\delta (R^2 + \ell^2)}
\]
\[
\frac{1}{\ell^2} \ll \frac{1}{\delta^2}
\]

messy, messy.

\[
\frac{V_0^2}{\ell} \left( \frac{\partial V_r}{\partial \ell} + V_\ell \frac{\partial V_r}{\partial \ell} \right) = -\frac{1}{\rho \ell^2} \frac{\partial P^*}{\partial \ell} + V \frac{V_0}{\ell^2} \left( \frac{1}{R^2 + \ell^2} \frac{\partial}{\partial r} \left( (R^2 + \ell^2) \frac{\partial V_r}{\partial \ell} \right) - \frac{V_\ell}{(R^2 + \ell^2)^2} \right)
\]
\[
\text{Same size}
\]

Assume these are same size
\[
P^* = \rho V_0^2 \left( \frac{\delta^2}{\ell^2} \right) \left\{ \text{Small} \right. \]
$P^*$ is also size of pressure variation in $x$ because no pressure variation outside of boundary layer (at $y = \infty$ is constant)

$Z$-momentum:

$$V_r \frac{\partial v_r}{\partial r} + V_z \frac{\partial v_z}{\partial z} = -\frac{\partial P}{\partial r} + V \left( \frac{1}{\rho} \frac{\partial \rho}{\partial r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right)$$

$$\frac{V_\infty^2}{S} \frac{\partial^2 v_z}{\partial r^2} + \frac{V_\infty^2}{L} \frac{\partial v_z}{\partial z} = \frac{P^*}{\rho L} \frac{\partial \theta}{\partial z} + V \left( \frac{1}{\theta} \frac{\partial \theta}{\partial r} \frac{\partial (R+f)}{\partial r} \left( \frac{\partial v_z}{\partial r} \right) \right)$$

$$\frac{V_\infty^2}{L} \frac{\partial v_r}{\partial r} + V_\infty \frac{\partial^2 v_r}{\partial z^2} = \frac{V_\infty}{\rho L} \frac{\partial \rho}{\partial z} \left( \frac{1}{\theta+f} \frac{\partial (R+f)}{\partial r} \frac{\partial v_z}{\partial r} \right)$$

Some size

much smaller than convective term

$$\frac{V_\infty^2}{L} \ll \frac{V_\infty}{L} \frac{\partial \theta}{\partial z}$$

$$\frac{V_\infty}{L} \ll \frac{\partial \rho}{\partial z}$$

$$\frac{\partial \theta}{\partial r} \frac{\partial (R+f)}{\partial r} \frac{\partial v_z}{\partial r}$$

$$\frac{\partial^2 v_z}{\partial z^2}$$

$$\frac{\partial \rho}{\partial z}$$

$$\frac{1}{\theta+f} \frac{\partial (R+f)}{\partial r}$$

should be same size as convective

So

$$\delta \sim \frac{1}{L} \left( \frac{1}{\theta} \right)$$
\[
\frac{d^2 f}{dr^2} = \left( \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{df}{dr} \right) \right) \frac{1}{r^2} \\
R^n \text{ is constant so can bring out of derivative}
\]
7.15 Discuss whether fully developed laminar incompressible flow between parallel plates, Eq. (4.143) and Fig. 4.16b, represents an exact solution to the boundary-layer equations (7.19) and the boundary conditions (7.20). In what sense, if any, are duct flows also boundary-layer flows?

Solution: The analysis for flow between parallel plates leads to Eq. (4.143):

\[ u = \left( \frac{dp}{dx} \right) \frac{h^2}{2\mu} \left( 1 - \frac{y^2}{h^2} \right); \quad v = 0; \quad \frac{dp}{dx} = \text{constant} < 0; \quad \frac{dp}{dy} = 0, \quad u(\pm h) = 0 \]

It is indeed a “boundary layer,” with \( v \ll u \) and \( \partial p/\partial y \approx 0 \). The “freestream” is the centerline velocity, \( u_{\text{max}} = (-dp/dx)(h^2/2\mu) \). The boundary layer does not grow because it is constrained by the two walls. The entire duct is filled with boundary layer. Ans.

7.16 A thin flat plate 55 by 110 cm is immersed in a 6-m/s stream of SAE 10 oil at 20°C. Compute the total friction drag if the stream is parallel to (a) the long side and (b) the short side.

Solution: For SAE 30 oil at 20°C, take \( \rho = 891 \text{ kg/m}^3 \) and \( \mu = 0.29 \text{ kg/m} \cdot \text{s} \).

(a) \( L = 110 \text{ cm}, \quad \text{Re}_L = \frac{891(6.0)(1.1)}{0.29} = 20300 \text{ (laminar)}, \quad C_D = \frac{1.328}{(20300)^{1/2}} \approx 0.00933 \)

\[ F = C_D \left( \frac{\rho}{2} \right) U^2 (2bL) = 0.00933 \left( \frac{891}{2} \right)(6)^2 (2)(0.55)(1.1) \approx 181 \text{ N} \quad \text{Ans. (a)} \]

The drag is 41% more if we align the flow with the short side:

(b) \( L = 55 \text{ cm}, \quad \text{Re}_L = 10140, \quad C_D = 0.0132, \quad F = 256 \text{ N} \quad \text{(41% more)} \quad \text{Ans. (b)} \)

7.17 Helium at 20°C and low pressure flows past a thin flat plate 1 m long and 2 m wide. It is desired that the total friction drag of the plate be 0.5 N. What is the appropriate absolute pressure of the helium if \( U = 35 \text{ m/s} \)?
Problem 7:

\[ \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

\[ \hat{x} = \frac{x}{l}, \quad \hat{y} = \frac{y}{l}, \quad \hat{u} = \frac{u}{u_\infty}, \quad \hat{v} = \frac{v}{u_\infty} \]

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial \hat{x}} \frac{\partial}{\partial x} = \frac{1}{l} \frac{\partial}{\partial \hat{x}} \]

\[ \frac{\partial}{\partial y} = \frac{\partial}{\partial \hat{y}} \frac{\partial}{\partial y} = \frac{1}{l} \frac{\partial}{\partial \hat{y}} \]

\[ \rho \left( u_\infty \hat{u} \frac{1}{l} \frac{\partial u_\infty \hat{u}}{\partial \hat{x}} + u_\infty \hat{v} \frac{1}{l} \frac{\partial u_\infty \hat{v}}{\partial \hat{y}} \right) = -\frac{P^*}{l} \frac{\partial \hat{P}}{\partial \hat{x}} + \frac{u_\infty}{l^2} \left( \frac{\partial^2 u_\infty}{\partial \hat{x}^2} + \frac{\partial^2 u_\infty}{\partial \hat{y}^2} \right) \]

\[ \frac{P^*}{l} \left( \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} \right) = -\frac{P^*}{l} \frac{\partial \hat{P}}{\partial \hat{x}} + \frac{u_\infty}{l^2} \left( \frac{\partial^2 u_\infty}{\partial \hat{x}^2} + \frac{\partial^2 u_\infty}{\partial \hat{y}^2} \right) \]

---

Compare These

\[ \frac{\rho u_\infty^2}{l} = \frac{\rho u_\infty l}{\mu} = Re \]