HW6

1. Book problems 8.5, 8.6, 8.9, 8.23, 8.31

2. Add an equal strength sink and a source separated by a small distance, dx, and take the limit of dx approaching zero to obtain the following equations for a doublet

$$\psi = -\frac{\lambda \mathrm{Sin}\theta}{r}, \quad \phi = \frac{\lambda \mathrm{Cos}\theta}{r}$$

3. Use the given MATLAB code (posted on the web) and evaluate the following flow fields
    a. A sink at point (0.1, 0) with strength of -3 plus a source at point (-0.1, 0) with strength 3
    b. Rankine Half Body: A source at point (-0.2, 0) with strength of 2 plus a uniform flow of strength 20 m/s
    c. Rankine Oval: A source at point (-0.2, 0) with strength of 2, a sink at point (0.3, 0) with strength of 2, a uniform flow of strength 20 m/s.
    d. Add a vortex of strength 1 located at point (0.05, 0) to part c.

4. For a cylindrical surface surrounding the flow field in part (d) of problem 3 calculate the force on this surface if the surface is located at r=50.

Solution: Evaluation of the laplacian of (*1/r*) shows that it is *not* legitimate:

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right] = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{1}{r^2}\right)\right] = \frac{\mathbf{1}}{\mathbf{r^3}} \neq \mathbf{0} \quad \textbf{Illegitimate} \quad \textit{Ans.}$$

8.5 Consider the two-dimensional velocity distribution u = –By, v = +Bx, where B is a constant. If this flow possesses a stream function, find its form. If it has a velocity potential, find that also. Compute the local angular velocity of the flow, if any, and describe what the flow might represent.

Solution: It <u>does</u> has a stream function, because it satisfies continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + 0 = 0 \text{ (OK);} \quad \text{Thus } u = -By = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = Bx = -\frac{\partial \psi}{\partial x}$$

$$\text{Solve for} \quad \psi = -\frac{\mathbf{B}}{\mathbf{2}}(\mathbf{x^2} + \mathbf{y^2}) + \mathbf{const} \quad \textit{Ans.}$$

It does <u>not</u> have a velocity potential, because it has a non-zero curl:

$$2\omega = \text{curl } \mathbf{V} = \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \mathbf{k}[B - (-B)] = 2B\mathbf{k} \neq 0 \quad \text{thus } \phi \textbf{ does not exist} \quad \textit{Ans.}$$

The flow represents <u>solid-body rotation</u> at uniform clockwise angular velocity *B*.

8.6 If the velocity potential of a realistic two-dimensional flow is $\phi = C\ln(x^2 + y^2)^{1/2}$, where C is a constant, find the form of the stream function $\psi(x, y)$. *Hint*: Try polar coordinates.

Solution: Using polar coordinates is certainly an excellent hint! Then the velocity potential translates simply to $\phi = C \ln(r)$, which is a line source. Equation (8.12*b*) also shows that,

$$\text{Eq. (8.12}b\text{):} \quad \psi = C\theta = \mathbf{C}\,\mathbf{tan^{-1}}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) \quad \textit{Ans.}$$

8.7 Consider a flow with constant density and viscosity. If the flow possesses a velocity potential as defined by Eq. (8.1), show that it exactly satisfies the full Navier-Stokes equation (4.38). If this is so, why do we back away from the full Navier-Stokes equation in solving potential flows?

Solution: If $\mathbf{V} = \nabla\phi$, the full Navier-Stokes equation is satisfied identically:

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{V} \quad \text{becomes}$$

$$\rho\left[\nabla\left(\frac{\partial\phi}{\partial t}\right) + \nabla\left(\frac{V^2}{2}\right)\right] = -\nabla p - \nabla(\rho g z) + \mu\nabla(\nabla^2\phi), \text{ where the last term is } \mathbf{zero}.$$

The viscous (final) term drops out identically for potential flow, and what remains is

$$\frac{\partial\phi}{\partial t} + \frac{V^2}{2} + \frac{p}{\rho} + gz = \text{constant} \quad (\textbf{Bernoulli's equation})$$

The Bernoulli relation is an exact solution of Navier-Stokes for potential flow. We don't exactly "back away," we need also to solve $\nabla^2\phi = 0$ in order to find the velocity potential.

8.8 For the velocity distribution of Prob. 8.5, u = –By, v = +Bx, evaluate the circulation Γ around the rectangular closed curve defined by (x, y) = (1, 1), (3, 1), (3, 2), and (1, 2).

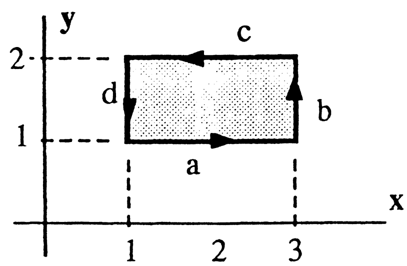


Fig. P8.8

Solution: Given $\Gamma = \int \mathbf{V}\cdot d\mathbf{s}$ around the curve, divide the rectangle into (a, b, c, d) pieces as shown:

$$\Gamma = \int_a u\,ds + \int_b v\,ds + \int_c u\,ds + \int_d v\,ds = (-B)(2) + (3B)(1) + (2B)(2) + (-B)(1)$$

$$\text{or} \quad \Gamma = \mathbf{+4B} \quad \textit{Ans.}$$

Since, from Prob. 8.5, $|\text{curl } \mathbf{V}| = 2B$, also $\Gamma = |\text{curl } \mathbf{V}| A_{region} = (2B)(2) = 4B$. (Check)

8.9 Consider the two-dimensional flow u = –Ax, v = +Ay, where A is a constant. Evaluate the circulation Γ around the rectangular closed curve defined by (x, y) = (1, 1), (4, 1), (4, 3), and (1, 3). Interpret your result especially *vis-a-vis* the velocity potential.

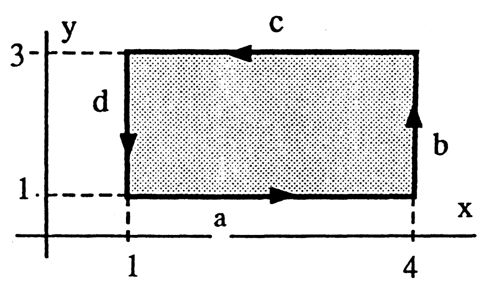


Fig. P8.9

Solution: This pattern is the same as Fig. 8.6 in the text, except it is upside down. There is a stagnation point at (x, y) = (0, $-K/U$). *Ans.*

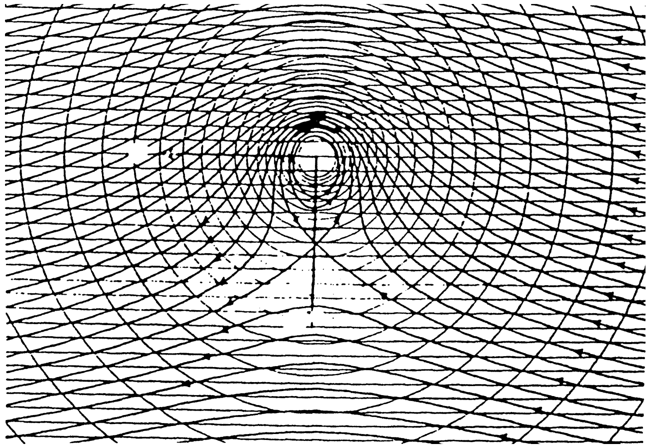


Fig. P8.22

8.23 Find the resultant velocity vector induced at point A in Fig. P8.23 due to the combination of uniform stream, vortex, and line source.

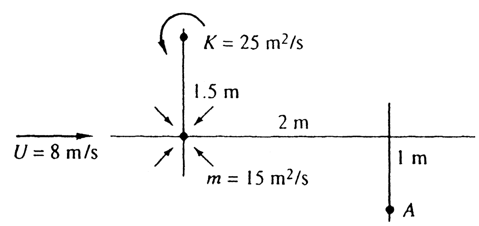
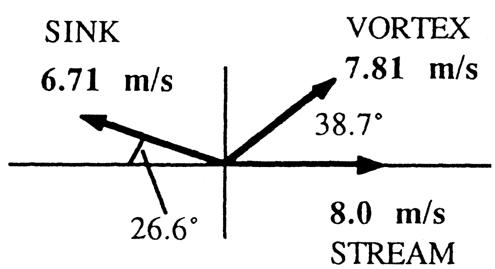


Fig. P8.23

Solution: The velocities caused by each term—stream, vortex, and sink—are shown at right. They have to be added together vectorially to give the final result:



$$\mathbf{V} = \mathbf{11.3}\ \frac{\mathbf{m}}{\mathbf{s}} \quad \text{at } \theta = \mathbf{44.2°} \ \angle \quad \textit{Ans.}$$

8.24 Line sources of equal strength $m = Ua$, where U is a reference velocity, are placed at $(x, y) = (0, a)$ and $(0, -a)$. Sketch the stream and potential lines in the upper half plane. Is $y = 0$ a "wall"? If so, sketch the pressure coefficient

$$C_p = \frac{p - p_0}{\frac{1}{2}\rho U^2}$$

along the wall, where p_0 is the pressure at (0, 0). Find the minimum pressure point and indicate

8.31 A Rankine half-body is formed as shown in Fig. P8.31. For the conditions shown, compute (a) the source strength m in m^2/s; (b) the distance a; (c) the distance h; and (d) the total velocity at point A.

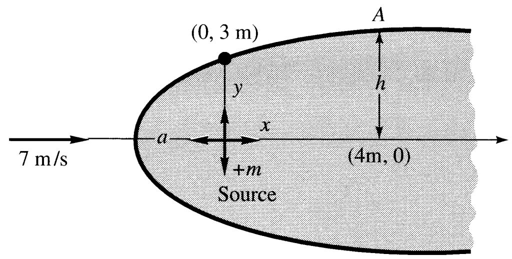


Fig. P8.31

Solution: The vertical distance above the origin is a known multiple of m and a:

$$y_{x=0} = 3\text{ m} = \frac{\pi m}{2U} = \frac{\pi m}{2(7)} = \frac{\pi a}{2},$$

$$\text{or}\quad \boldsymbol{m \approx 13.4\ \frac{m^2}{s}} \quad\text{and}\quad \boldsymbol{a \approx 1.91\text{ m}} \quad \textit{Ans. (a, b)}$$

The distance h is found from the equation for the body streamline:

$$\text{At}\quad x = 4\text{ m},\quad r_{body} = \frac{m(\pi-\theta)}{U\sin\theta} = \frac{13.4(\pi-\theta)}{7\sin\theta} = \frac{4.0}{\cos\theta},\quad \text{solve for } \theta \approx 47.8°$$

$$\text{Then}\quad r_A = 4.0/\cos(47.8°) = 5.95\text{ m} \quad\text{and}\quad \mathbf{h} = r\sin\theta \approx \mathbf{4.41\ m} \quad \textit{Ans. (c)}$$

The resultant velocity at point A is then computed from Eq. (8.18):

$$V_A = U\left(1+\frac{a^2}{r^2}+\frac{2a}{r}\cos\theta_A\right)^{1/2} = 7\left[1+\left(\frac{1.91}{5.95}\right)^2+2\left(\frac{1.91}{5.95}\right)\cos 47.8°\right]^{1/2} \approx \mathbf{8.7\ \frac{m}{s}} \quad \textit{Ans. (d)}$$

8.32 Sketch the streamlines, especially the body shape, due to equal line sources m at (–a, 0) and (+a, 0) plus a uniform stream $U_\infty = ma$.

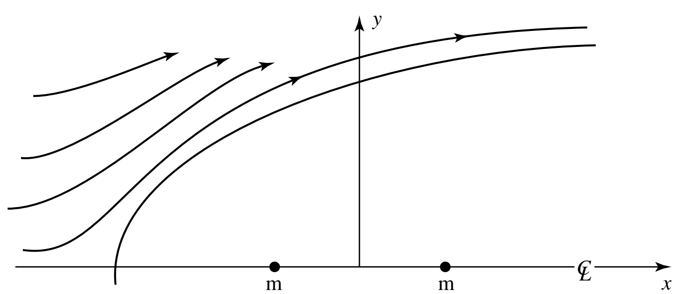


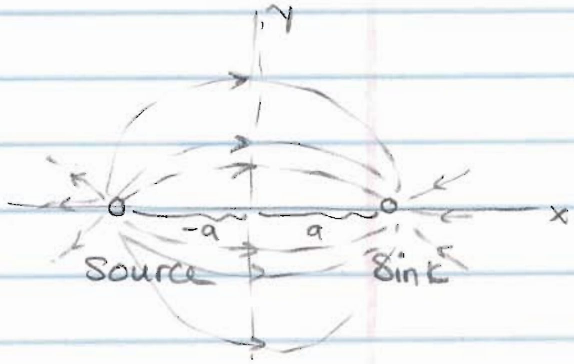
Fig. P8.32

Solution: As shown, a half-body shape is formed quite similar to the Rankine half-body. The stagnation point, for this special case $U_\infty = ma$, is at $x = (-1 - \sqrt{2})a = -2.41a$. The half-body shape would vary with the dimensionless source-strength parameter ($U_\infty a/m$).

Problem 2 – Doublet

$\underline{a = dx}$

Superposition Sink + Source



$$\psi = \psi_{Source} + \psi_{Sink}$$

$$= m\theta_1 - m\theta_2 = m(\theta_1 - \theta_2) = m\left[\tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x+a}\right]$$

Knowing that

$$\tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

then $$\psi = -m\tan^{-1}\left[\frac{2ay}{x^2+y^2-a^2}\right] \quad (1)$$

$$\phi = \phi_{Source} + \phi_{Sink}$$

$$= m\ln r_1 - m\ln r_2 = m\ln\frac{r_1}{r_2}$$

or

$$\phi = \frac{m}{2}\ln\left\{\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right\} \quad (2)$$

now let $a \to 0$

From (1),

$$\psi = \lim_{a \to 0} \left[-m \tan^{-1} \frac{2ay}{x^2+y^2-a^2} \right]$$

let's ignore the 2nd order terms

$$\psi = \lim_{a \to 0} -m \tan^{-1} \frac{2ay}{x^2+y^2}$$

if $a \to 0$ then $\frac{2ay}{x^2+y^2}$ becomes a very small number and we know $\tan^{-1}\theta \simeq \theta$ if θ is very small, then

Ans.

$$\psi \simeq -m \frac{2ay}{x^2+y^2} = -\frac{\lambda y}{x^2+y^2} = \frac{-\lambda \sin\theta}{r}$$

$\lambda = 2am$

lets take limit of (2)

$$\phi = \lim_{a \to 0} \frac{m}{2} \ln \left\{ \frac{(x+a)^2+y^2}{(x-a)^2+y^2} \right\}$$

lets expand $(x \mp a)^2$ and ignore the 2nd Droder terms, then

$$\phi \approx \frac{m}{2} \ln \frac{x^2+y^2+2ax}{x^2+y^2-2ax} = \frac{m}{2} \ln \frac{r^2+2ax}{r^2-2ax}$$

now lets find this limit, but we need to Simplify first, from math we know that

$$\frac{1+\varepsilon}{1-\varepsilon} = 1+2\varepsilon+ \cdots \qquad |\varepsilon|<1 \qquad (3)$$

I can then rewrite

$$\frac{r^2+2ax}{r^2-2ax} = \frac{r^2(1+2ax/r^2)}{r^2(1-2ax/r^2)} = \frac{1+\frac{2ax}{r^2}}{1-\frac{2ax}{r^2}}$$

as $a \to 0$ the $\frac{2ax}{r^2}$ term is small

and I can replace it by the first two terms from the taylor series in (3)

$$\frac{1+\frac{2ax}{r^2}}{1-\frac{2ax}{r^2}} \simeq 1+2\left(\frac{2ax}{r^2}\right)$$

then

$$\phi = \frac{m}{2} \ln\left(1 + \frac{2(2ax)}{r^2}\right)$$

again we need to write the Taylor series exansion for the $\underline{\underline{\ln}}$ function we have

$$\ln(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} + \cdots \qquad |\varepsilon|<1$$

lets use up to the first order term

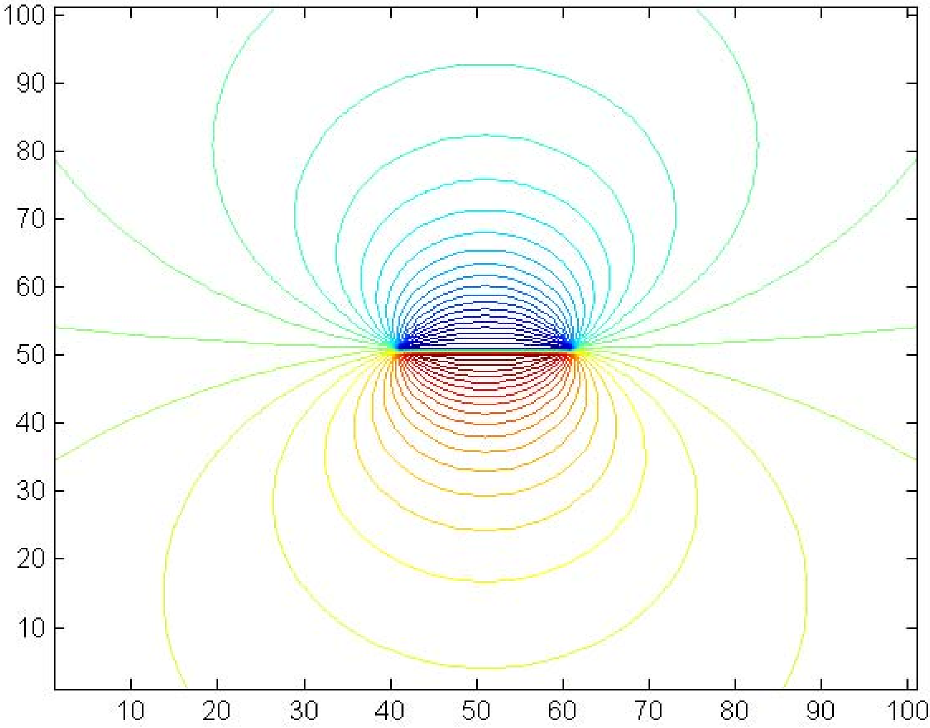
then $\varepsilon = \dfrac{2(2ax)}{r^2}$

$$\phi = \frac{m}{2}\left[\frac{2(2ax)}{r^2}\right] = \frac{2amx}{r^2} = \frac{\lambda x}{r^2}$$

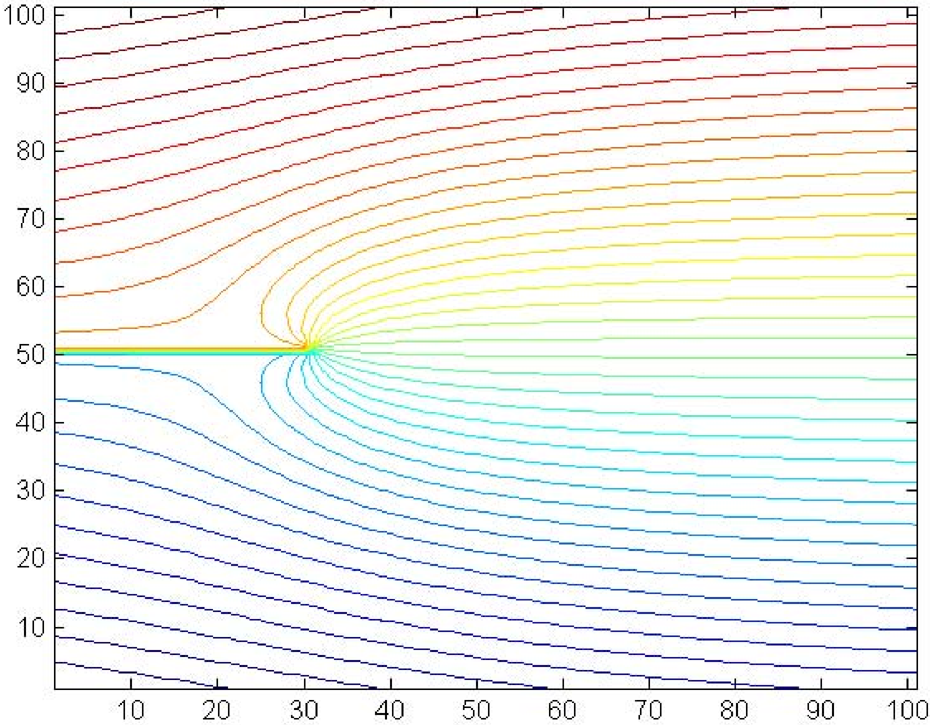
$$\text{or } \phi = \frac{\lambda x}{x^2+y^2} = \frac{\lambda \cos\theta}{r^2} \qquad \text{Ans.}$$

$\lambda = 2am$

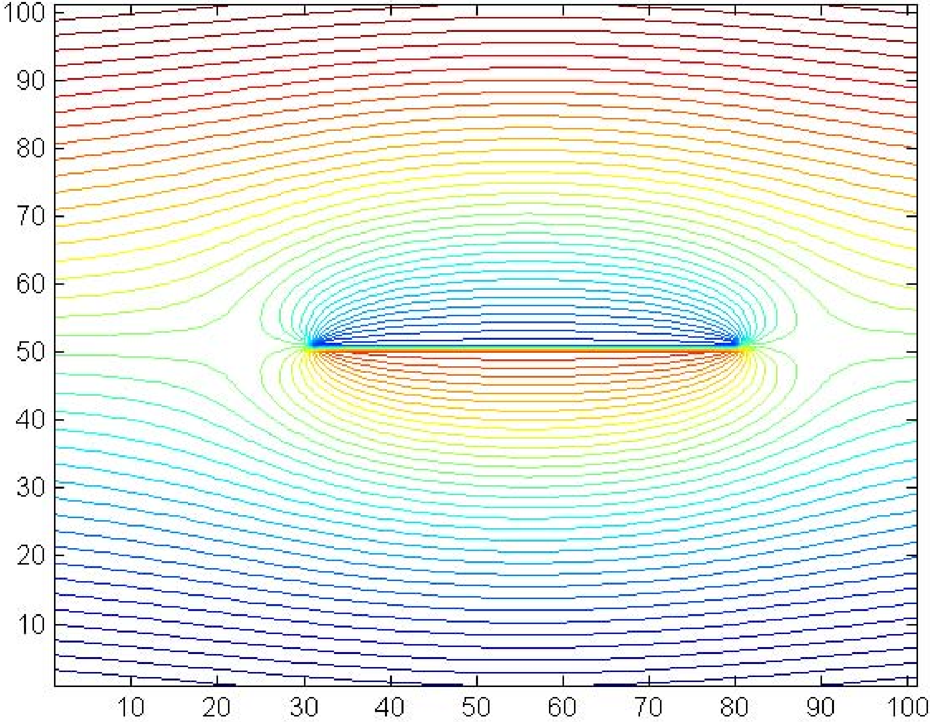
3-a



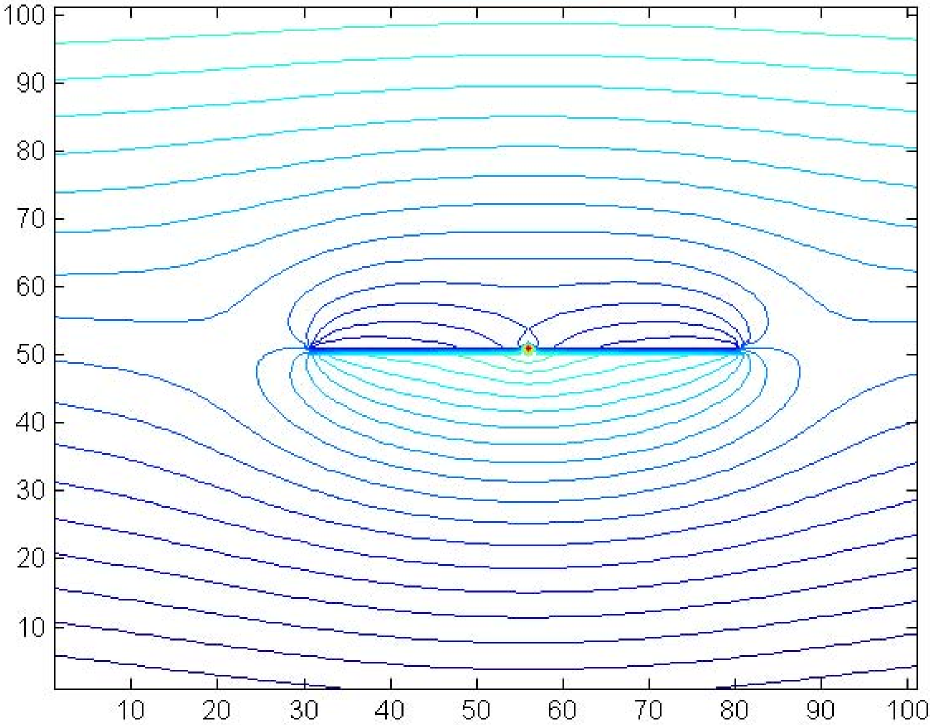
3-b



3-c



3-d



Problem 4.

$$\phi = \phi_{sink} + \phi_{source} + \phi_{vortex} + \phi_{stream}$$

$$= m_1 \ln r_1 + m_2 \ln r_2 - K\theta_3 + U_\infty r \cos\theta$$

where

$$r_1^2 = (x - x_1)^2 + (y - y_1)^2 = (x - 0.3)^2 + y^2$$

or

$$r_1^2 = x^2 + y^2 + 0.09 - 0.6x$$

$$\text{or } r_1^2 = r^2 + 0.09 - 0.6 r\cos\theta$$

we need this part to have r_1 as a function of to differenciate and find velocity components.

Similarly

$$r_2^2 = r^2 + 0.04 + 0.4 r\cos\theta$$

and

$$\theta_3 = \tan^{-1} \frac{y - y_3}{x - x_3} = \tan^{-1} \frac{y}{x - 0.05}$$

or

$$\theta_3 = \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 0.05}\right)$$

now lets find u_θ $(m_1 = 2, m_2 = 2, K = 1, U_\infty = 20)$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[2 \ln \frac{r_2}{r_1} - \theta_3 + 20 r\cos\theta\right]$$

See the Maple/Matlab codes for the rest.

Problem 4

$$m1 := -2$$

$$m2 := 2$$

$$K := 1$$

$$Uinf := 20$$

$$r1 := \sqrt{r^2 + 0.09 - 0.6\, r \cos(\theta)}$$

$$r2 := \sqrt{r^2 + 0.04 + 0.4\, r \cos(\theta)}$$

$$\theta 3 := \arctan\left(\frac{r \sin(\theta)}{r \cos(\theta) - 0.05}\right)$$

$$\phi := -\ln(r^2 + 0.09 - 0.6\, r \cos(\theta)) + \ln(r^2 + 0.04 + 0.4\, r \cos(\theta)) - \arctan\left(\frac{r \sin(\theta)}{r \cos(\theta) - 0.05}\right) + 20\, r \cos(\theta)$$

$$vt := \frac{1}{r}\left(-\frac{0.6\, r \sin(\theta)}{r^2 + 0.09 - 0.6\, r \cos(\theta)} - \frac{0.4\, r \sin(\theta)}{r^2 + 0.04 + 0.4\, r \cos(\theta)} - \frac{\frac{r \cos(\theta)}{r \cos(\theta) - 0.05} + \frac{r^2 \sin(\theta)^2}{(r \cos(\theta) - 0.05)^2}}{1 + \frac{r^2 \sin(\theta)^2}{(r \cos(\theta) - 0.05)^2}} - 20\, r \sin(\theta)\right)$$

$$vr := -\frac{2r - 0.6\cos(\theta)}{r^2 + 0.09 - 0.6\,r\cos(\theta)} + \frac{2r + 0.4\cos(\theta)}{r^2 + 0.04 + 0.4\,r\cos(\theta)} - \frac{\dfrac{\sin(\theta)}{r\cos(\theta) - 0.05} - \dfrac{r\sin(\theta)\cos(\theta)}{(r\cos(\theta) - 0.05)^2}}{1 + \dfrac{r^2\sin(\theta)^2}{(r\cos(\theta) - 0.05)^2}} + 20\cos(\theta)$$

$$r := 50.$$

$$V = \sqrt{V_r^2 + V_\theta^2}$$

$$p = \rho\left(V^2 - V_\infty^2\right)$$

$$Drag = \int_0^{2\pi} pCos\theta(r.b.d\theta)$$

$$Lift = \int_0^{2\pi} pSin\theta(r.b.d\theta)$$

$$Drag := -0.00006288218279\, b\, \rho$$

$$Lift := -62.83310971\, b\, \rho$$