Data-Driven Dynamical Systems.

about $x'=f(x,t)$

On Koopman Operator – On DMD

For a spatiotemporal process -

POD is the best basis

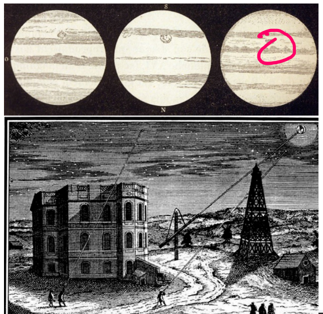
- – best measured in most energy successively per mode, in time average
- =fastest decaying time average power spectrum
- =usually the best approximation when truncating that finite series approximation to a few terms

DMD is the best description of the process

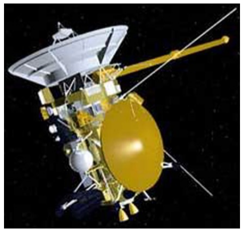
- -best in that it looks most like a simple process
- -looks like the linear process
- -describes even a nonlinear system as having connections to a linear one
- -so Fourier applies in the interpretation and the modes are “interesting”

Intermission of Fun Glitz Things Slides - Lots of Great Information to exploit in eigenfunctions and values

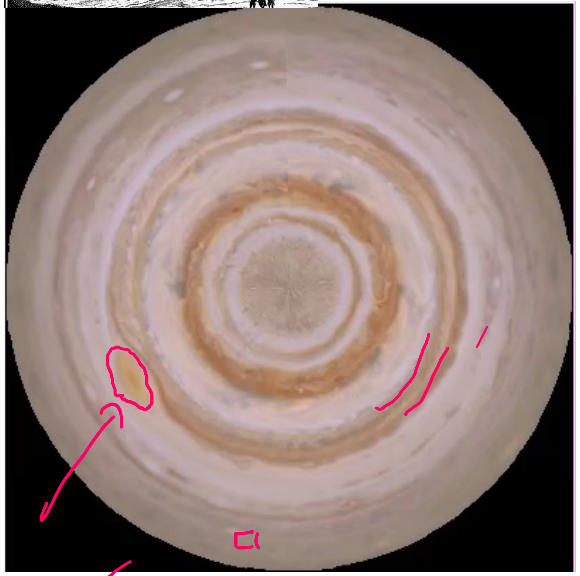
Coherence for Example

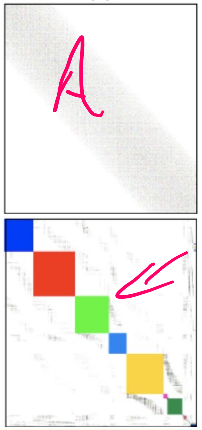


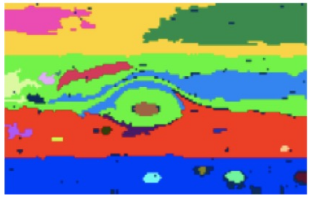
Cassini in Paris in the 1660's.

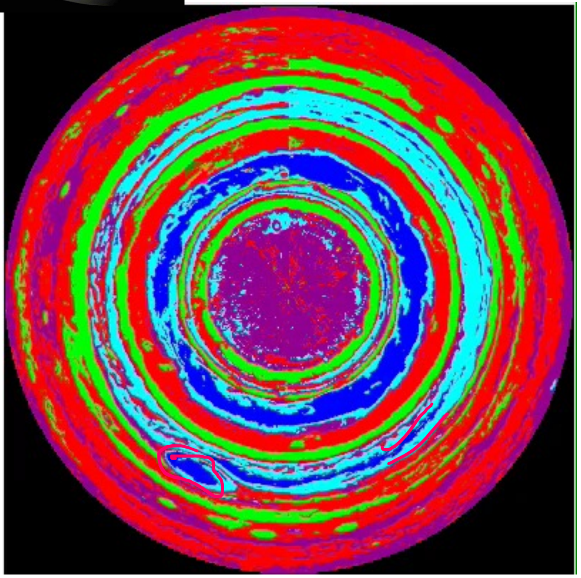


Cassini "today", Dec 2000





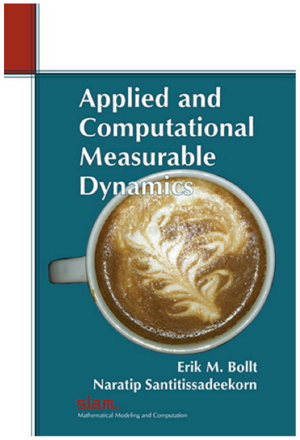


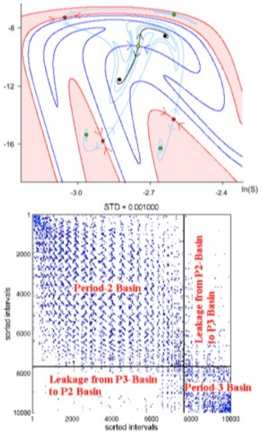


Spectral partitioning of transfer operators by directed graph Laplacian following work of F. Chung – leaning on Raleigh-Ritz quotient concept

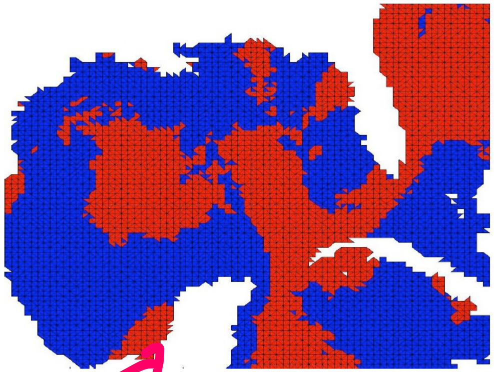
$B = PAP^T$

Eigenfunctions in the *South*

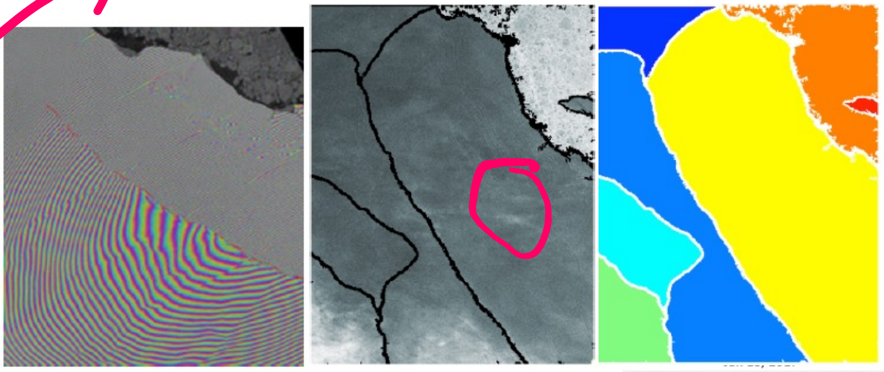


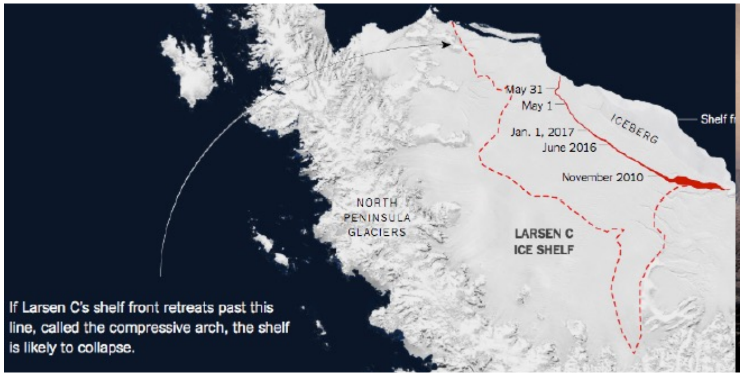


Epidemic bursting in childhood a disease.

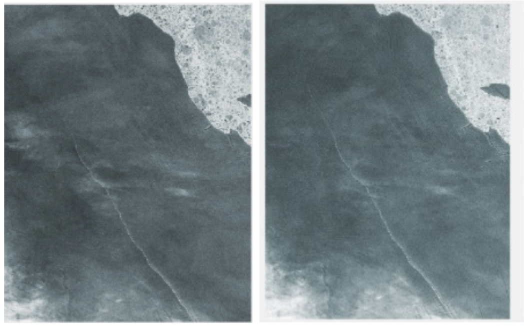


Deep Water Horizon, Gulf Oil disaster 2010





If Larsen C's shelf front retreats past this line, called the compressive arch, the shelf is likely to collapse.



Larcen C ice shelf, Antarctica, '17

Koopman Operator as Composition Operator

- $\dot{x} = F(x), \qquad F : R^d \to R^d$ -or-
- (semi-) flow $S_t : M \to M$ each $t \in M \subset R^d$
- Let $\Im$ be set of "observation" functions (measurable). E.g.

$$\Im = L^2(M) = \{g : \int_M |g|^2 d\mu < \infty\}$$

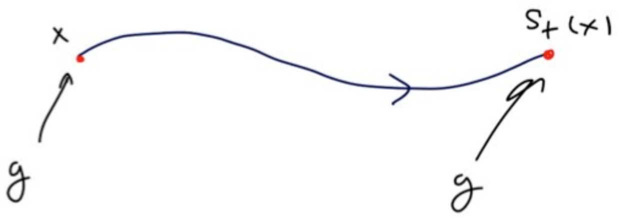
- Then let, $\mathrm{K}_{S_t}[g](x) = g \circ S_t(x)$

So **measure**/observe g not at x but **downstream** at $S_t(x)$

K is linear.

$$K[a_1 g_1 + a_2 g_2](x)$$

$$= a_1 K[g_1](x) + a_2 K[g_2](x)$$



$K : \Im \to \Im$

operator

Koopman is adjoint of Frobenius-Perron

- $< g, P[f] >_{\mathcal{F}^* x \mathcal{F}} = < K[g], \mathrm{f} >_{\mathcal{F}^* x \mathcal{F}}$
- $P[f](x) = \int \delta\big(x - S_t(y)\big) f(y) dy$

Vs. $K[g](x) = \int \delta\big(x - S_t(y)\big) g(x) dx$

An Eigenfunction of K satisfies

$Av = \lambda^t v$

$$K_t[g](x) = g \circ S_t(x) =$$

$b^t = e^{\lambda t}$

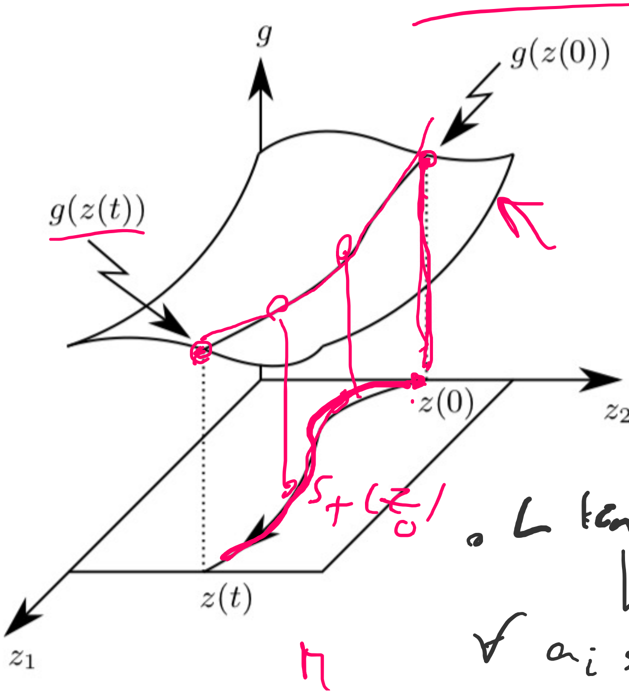
$$= b^t g(x) = e^{\lambda t} g(x)$$

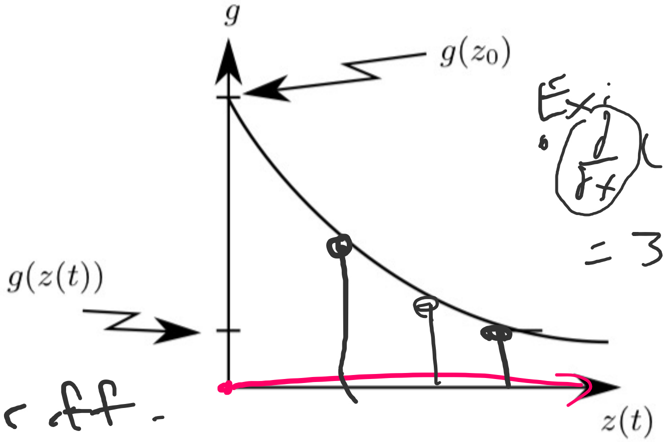
$v = Av$

eigenvalue λ **eigenfunction** g

K is linear operator even though it n have

$\int 3x^2 + 7x^3 dx = 3\int x^2 dx + 7\int x^3 dx$





Ex:

$\frac{\partial}{\partial x}(3x^2 + 7x)$

$= 3\frac{\partial}{\partial x}x^2 + 7\frac{\partial}{\partial x}x$

$= 3.2x + 7 \cdot 1$

- L linear iff.

$L(a_1 v_1 + a_2 v_2) = a_1 L(v_1) + a_2 L(v_2)$

∀ a_i scalar, v_i in vector space.

How to find eigenfunctions of **Koopman** operator of flow of $\dot{x} = f(x)$ → $\dot z = f(z)$

z

Solve $\boxed{\nabla g \cdot f = \lambda f}$

Why? Given, $\mathcal{K}_t[g](z) = e^{\lambda t} g(z)$ $\dot{g} = \lambda g.$ demand,

Or generally, $\dot{g} = d(g(z))/dt = \sum_i \frac{\partial g(z)}{\partial z_i} \dot{z}_i = \nabla g \cdot \dot{z} = \nabla g \cdot F(z).$

$$A_{\mathcal{K}} g(x) = \lim_{t \to 0} \frac{g(S_t(x_0)) - g(x_0)}{t} = \lim_{t \to 0} \frac{g(x(t)) - g(x_0)}{t},$$

which follows from the definition of the operator If g is continuously differentiable compact set $\mathbb{X}$, $g \in C^1(\mathbb{X})$, we can apply the mean value theorem to obtain

$$A_{\mathcal{K}} g(x) = \sum_{i=1}^{d} \frac{\partial g}{\partial x_i} F_i(x)$$
$$= \nabla g \cdot F(x).$$

$$S(z(t+1)) = g \circ S_t(z)$$

$$\dot g = \frac{\partial g}{\partial z} \cdot \cancel{\frac{\partial z}{\partial t}}$$

$$\nabla g \cdot F$$

$f(z)$

$$\frac{\partial g}{\partial z} = \frac{\partial g}{\partial z_i}\ \frac{\partial g_i}{\partial z_i} = \nabla g$$

Theorem *Given a domain $\mathbb{X} \subseteq M \subseteq \mathbb{R}^d$, $z \in \mathbb{X}$, and $\dot{z} = F(z)$ with $F : \mathbb{X} \to \mathbb{R}^d$, then the corresponding Koopman operator has eigenfunctions $g(z)$ that are solutions of the linear PDE,*

$$\nabla g \cdot F(z) = \lambda g(z), \tag{8}$$

if $\mathbb{X}$ is compact and $g(z) : \mathbb{X} \to \mathbb{C}$ is in $C^1(\mathbb{X})$, or alternatively, if $g(z)$ is $C^2(\mathbb{X})$.

where ϕ_k and λ_k are the eigenvectors and eigenvalues of $\mathbf{A}$ and $\mathbf{b}$ contains the coefficients of the initial condition $\mathbf{x}_1$ in the eigenvector basis so that $\mathbf{x}_1 = \mathbf{\Phi b}$. The DMD algorithm produces a low rank eigendecomposition of $\mathbf{A}$ that optimally fits the measured trajectory $\mathbf{x}_k$ for $k = 1, 2, ..., m$ in a least-squares sense so that

$$\|\mathbf{x}_{k+1} - Ax_k\|_2$$

$$\|X' - AX\|_F = \sum_{k=1}^{m-1} \|x_{k+1} - Ax_k\|_2$$

is minimized across all points for $k = 1, 2, ..., m-1$.

Given m snapshots of data, each of length n,

$$\mathbf{X} = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{m-1} \\ | & | & & | \end{bmatrix},$$

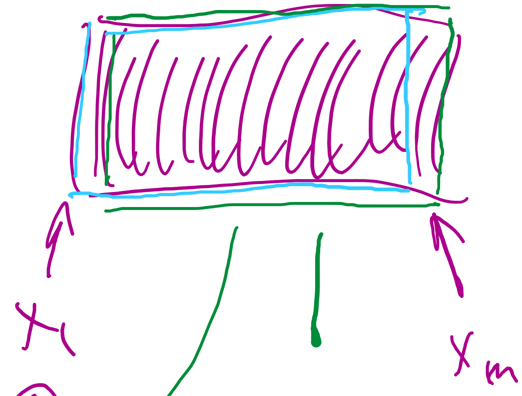
A A A

$$\mathbf{X}' = \begin{bmatrix} | & | & & | \\ \mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_m \\ | & | & & | \end{bmatrix}$$

input output

I have X, X'

But I dont know A



$X' = AX$?

$$\underline{\mathbf{X}'} \approx \underline{\mathbf{A}}\,\underline{\mathbf{X}}.$$

Ansatz

$$\underline{\|\mathbf{X}' - \mathbf{A}\mathbf{X}\|_F}$$

The best-fit A matrix is given by

$$\mathbf{A} = \mathbf{X}'\mathbf{X}^{\dagger}$$

where † denotes the Morse-Penrose pseudoinverse.

$$\mathbf{X} \approx \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$$

$\mathbf{U} \in \mathbb{C}^{n\times r}, \mathbf{\Sigma} \in \mathbb{C}^{r\times r}, \mathbf{V} \in \mathbb{C}^{m\times r}$ and r is the rank

$$\mathbf{A} = \mathbf{X}'\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^*$$

BUT CAREFUL! Inverse usually dne – use penrose pseudo inverse

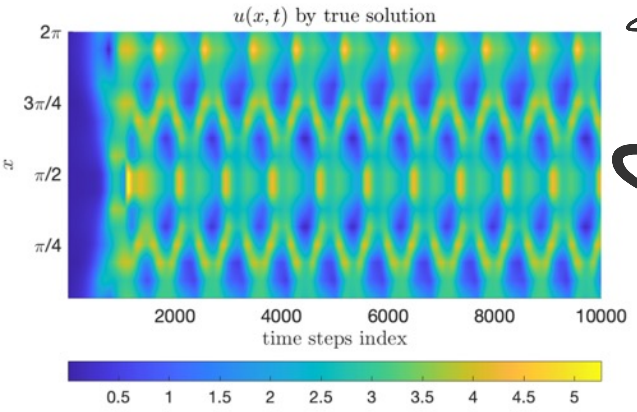
The left singular vectors $\mathbf{U}$ are POD modes.

On DMD (and the many variants) – (which these turn out to be some kind of estimator of Koopman).

Dynamic Mode Decomposition

Data Driven version of Koopman.

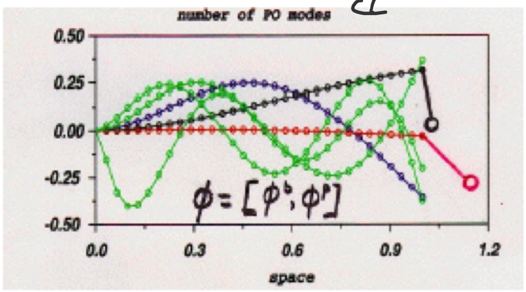
POD



DMD

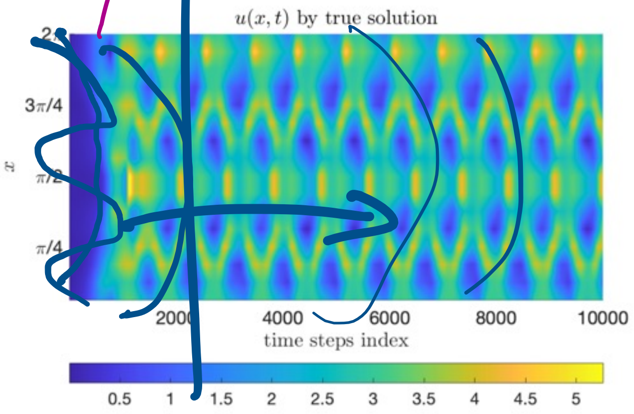
KS

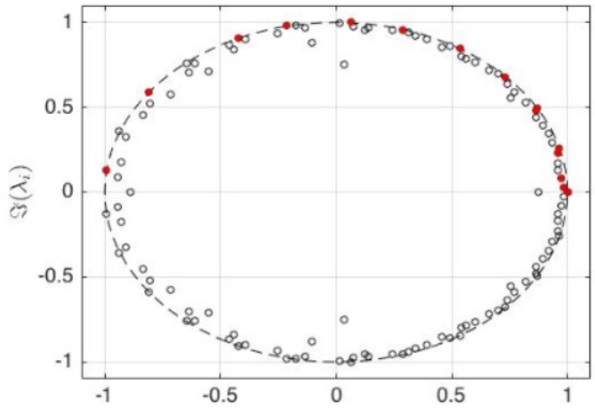
POD



$\phi = [\phi^b; \phi^p]$

But what does it mean? What do you get?





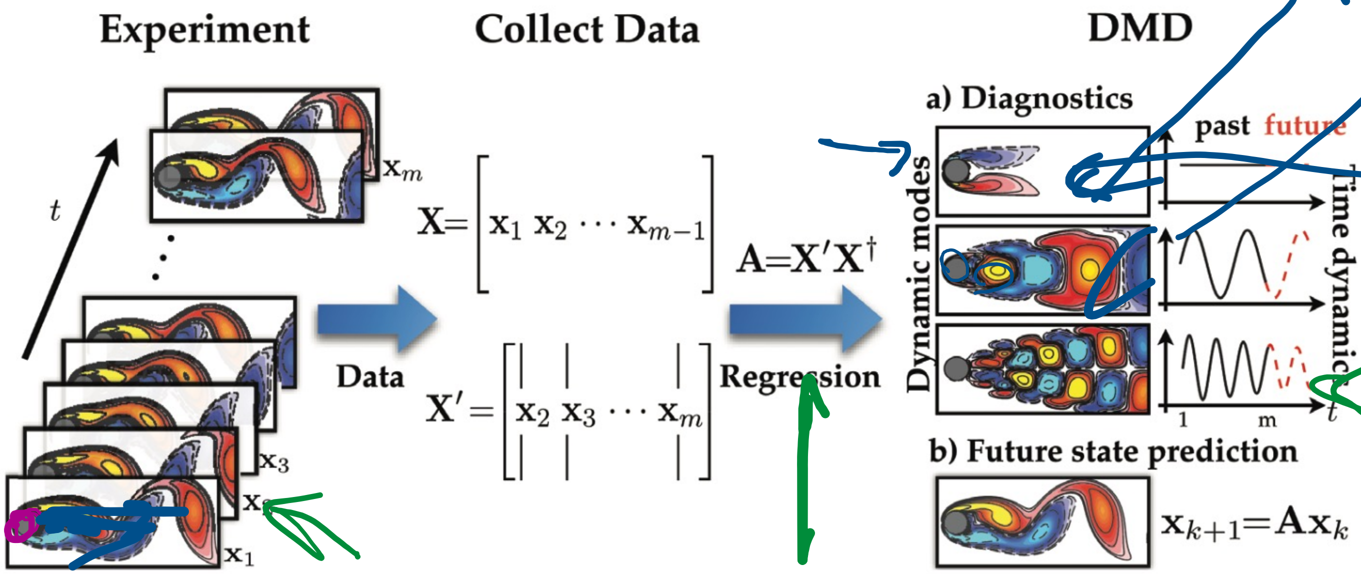
may be complex

ϕ_1

ϕ_2

$|\phi_1|$

$Re(\phi_1)$



Experiment

Collect Data

DMD

t

x_m

x_3

x_2

x_1

Data

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_{m-1} \end{bmatrix}$$

$$X' = \begin{bmatrix} | & | & & | \\ x_2 & x_3 & \cdots & x_m \\ | & | & & | \end{bmatrix}$$

$$A = X'X^{\dagger}$$

Regression

a) Diagnostics

Dynamic modes

past future

Time dynamics

1 m t

b) Future state prediction

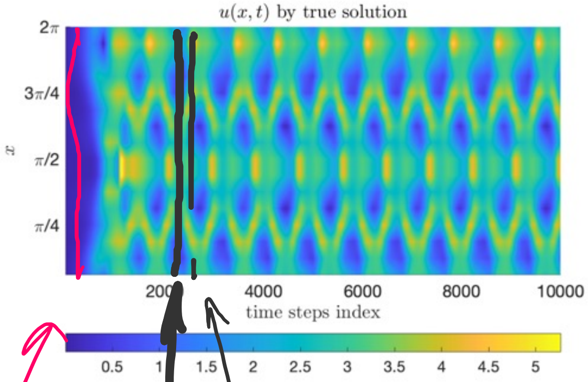
$$x_{k+1} = Ax_k$$

Given data collected from some dynamical system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t; \mu)$$

Big Matrix

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \quad , \vec{x}(0) = \vec{x}_0$$



With initial condition $\mathbf{x}(0)$, this system has

$$\mathbf{x}(t) = \sum_{k=1}^{n} \phi_k \exp(\omega_k t) b_k = \mathbf{\Phi} \exp(\mathbf{\Omega} t)\mathbf{b}$$

x_0

$x_i \quad t_{i+1}$

$\sim x(t_i)$

$$e^{\omega t} = e^{(a+ib)t} = e^{at} e^{ibt}$$

where $\boldsymbol{\phi}_k$ and ω_k are the eigenvectors and eigenvalues of A and the coefficients b_k are the coordinates of the initial condition $\mathbf{x}(0)$ in the eigenvector basis.

It is possible to describe a discrete time analog of the dynamical system above by taking time samples every Δt yielding,

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$$

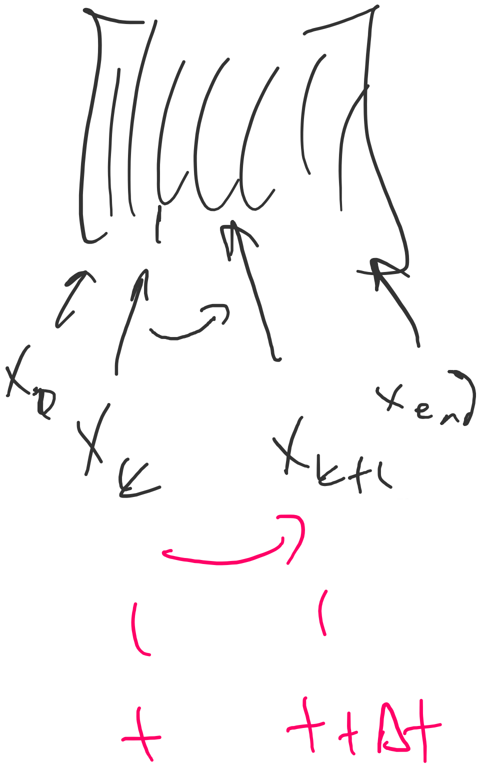
where

$$\mathbf{A} = \exp(\mathcal{A}\Delta t).$$

This system has the following solution:

$$\mathbf{x}_k = \sum_{j=1}^{r} \phi_j \lambda_j^k b_j = \mathbf{\Phi}\mathbf{\Lambda}^k \mathbf{b}$$

for each k.



$$\mathbf{A} = \mathbf{X}'\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^*$$

For computational efficiency, $\widetilde{\mathbf{A}}$, which is the $r \times r$ projection of the full matrix $\mathbf{A}$ onto POD modes, is typically used:

$$\widetilde{\mathbf{A}} = \mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{U}^*\mathbf{X}'\mathbf{V}\mathbf{\Sigma}^{-1}.$$

The matrix $\widetilde{\mathbf{A}}$ defines a low-dimensional linear model of the dynamical system on POD coordinates:

$$\widetilde{\mathbf{x}}_{k+1} = \widetilde{\mathbf{A}}\widetilde{\mathbf{x}}_k.$$

Next we compute the eigendecomposisiton of $\widetilde{\mathbf{A}}$:

$$\widetilde{\mathbf{A}}\mathbf{W} = \mathbf{W}\mathbf{\Lambda},$$

where the columns of $\mathbf{W}$ are the eigenvectors and $\mathbf{\Lambda}$ is a diagonal matrix containing the

The eigenvalues of A are given by diag$\mathbf{\Lambda}$ and the eigenvectors of $\mathbf{A}$ (the DMD modes) are given by the columns of $\mathbf{\Phi}$:

eigenvalues λ_k.

$$\mathbf{\Phi} = \mathbf{X}'\mathbf{V}\mathbf{\Sigma}^{-\mathbf{1}}\mathbf{W}.$$

The projected future solution can be given by the low-rank approximation to

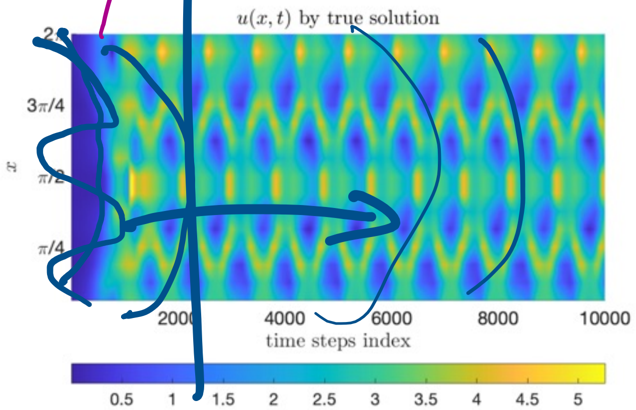
$$\mathbf{x}(t) = \sum_{k=1}^{r} \boldsymbol{\phi}_k \exp(\omega_k t) b_k = \mathbf{\Phi} \exp(\mathbf{\Omega} t)\mathbf{b}$$

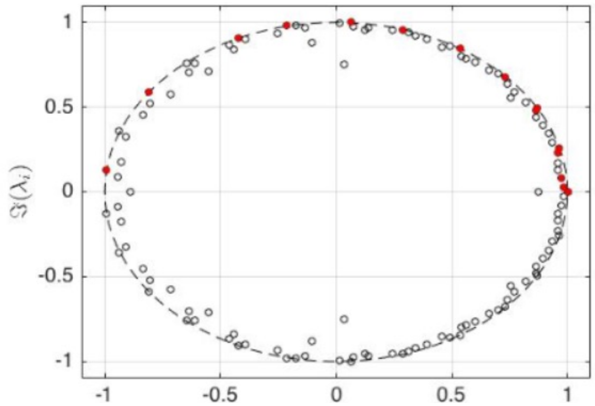
where $\omega_k = \ln(\lambda_k)/\Delta t$.

Recall that $\mathbf{b}$ contains the initial condition of the observable in the eigenvector basis so that $\mathbf{x}_1 = \mathbf{\Phi}\mathbf{b}$. $\mathbf{\Phi}$ need not be square so we use its pseudoinverse to find the vector $\mathbf{b}$:

$$\mathbf{b} = \mathbf{\Phi}^{\dagger}\mathbf{x}_1.$$

But what does it mean? What do you get?





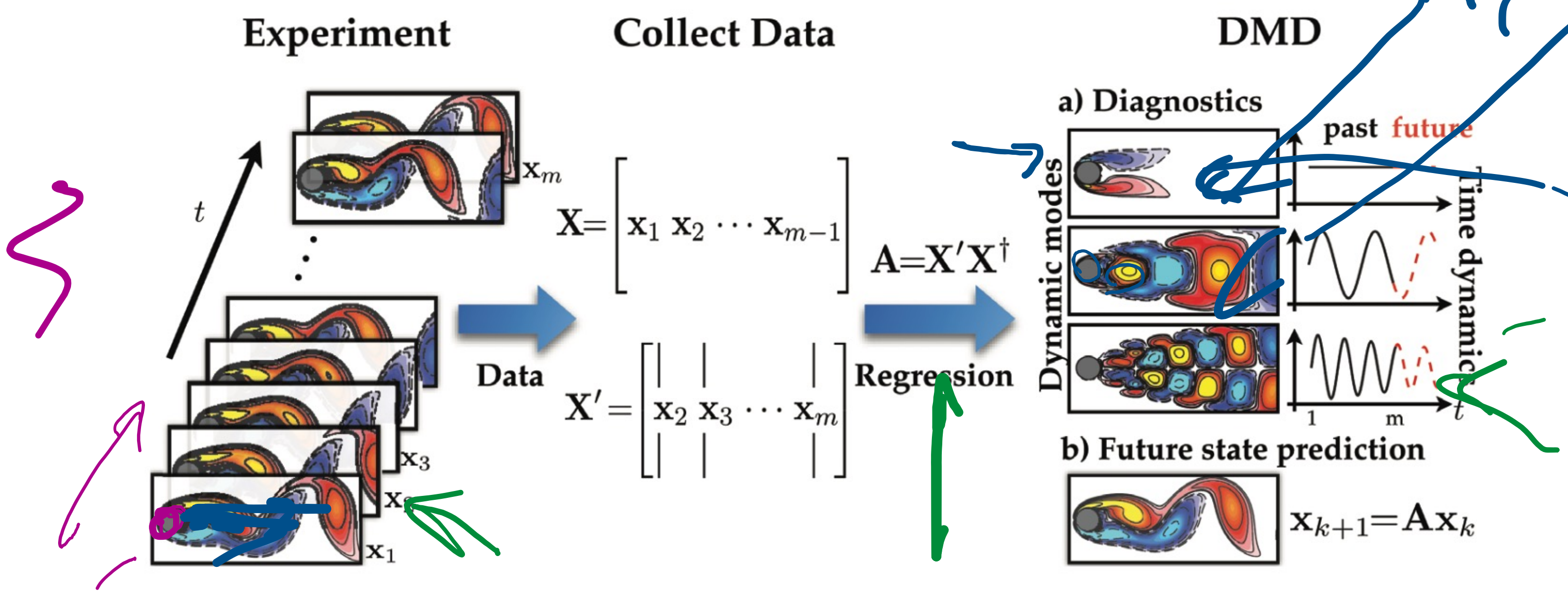
may be complex

ϕ_1

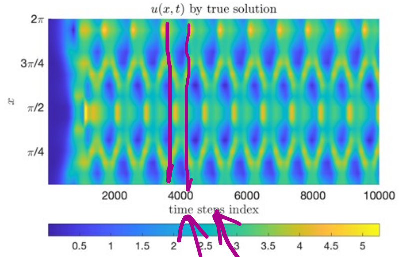
ϕ_2

$|\phi_1|$

$Re(\phi_1)$



Think VandermondeMatrix. - "the movie player"



ψ_{i+1}

ψ_i

$\psi_{i+1} = A\psi_i$

$$\Psi_0 := \left[\,\psi_0\ \psi_1 \cdots \psi_{N-1}\,\right] \in \mathbb{C}^{M\times N},$$

$$\Psi_1 := \left[\,\psi_1\ \psi_2 \cdots \psi_N\,\right] \in \mathbb{C}^{M\times N},$$

$$\begin{aligned}\Psi_1 &= \left[\,\psi_1\ \psi_2 \cdots \psi_N\,\right]\\ &= \left[\,A\,\psi_0 \quad A\,\psi_1 \quad \cdots \quad A\,\psi_{N-1}\,\right]\\ &= A\,\Psi_0.\end{aligned}$$

$$\psi_{t+1} \;=\; A\,\psi_t, \quad t \;=\; \{0,\ldots,N-1\}\,.$$

Koopman-DMD Modes

Van der Monde

$$\underbrace{\left[\,\psi_0 \quad \psi_1 \quad \cdots \quad \psi_{N-1}\,\right]}_{\Psi_0} \approx \underbrace{\left[\,\phi_1 \quad \phi_2 \quad \cdots \quad \phi_r\,\right]}_{\Phi} \underbrace{\begin{bmatrix}\alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_r\end{bmatrix}}_{D_\alpha \;:=\; \mathrm{diag}\{\alpha\}} \underbrace{\begin{bmatrix}1 & \mu_1 & \cdots & \mu_1^{N-1}\\ 1 & \mu_2 & \cdots & \mu_2^{N-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \mu_r & \cdots & \mu_r^{N-1}\end{bmatrix}}_{V_{\mathrm{and}}},$$

Ψ_0

DMD modes

$\vec{\alpha} = \mathrm{Diag}(D_\alpha)$

$\psi_0 = \Phi \cdot \vec{\alpha} \cdot 1$

solve for $\vec{\alpha}$

ampl. α

- Given Data $D = \{\psi_0, \psi_1, \ldots, \psi_N\}$, $\psi_i \in \mathbb{C}^M$
- Assume Ansatz $\psi_{t+1} = A\psi_t$ - solve for $\underline{A}$
- Convenience split data Data matrices. $\underset{M\times N}{\Psi_1} = \underset{N\times N}{A}\,\underset{M\times N}{\Psi_0}$

Data matrices

$$\Psi_0 = [\psi_0 | \psi_1 | \ldots | \psi_{N-1}] \in \mathbb{C}^{M\times N}$$

$$\Psi_1 = [\psi_1 | \psi_2 | \ldots | \psi_N]$$

$M\times N$

- Exact DMD Goal optimal representation $\boxed{A_{DMD}} \in \mathbb{C}^{r\times r}$ rank r approx of matrix A in $\{$the basis spanned by the POD Modes

$r < N$ - ROM - simplicity

- What does does that Meen to project $A^{r \times N}$ into POD basis?

- Presume Ψ_0 already demeaned. $\Psi_0 = \overset{\overset{U_r}{\downarrow}}{U} \Sigma \overset{\overset{\text{\& keep.}}{\downarrow}}{V^*}$

columns of U are POD basis.

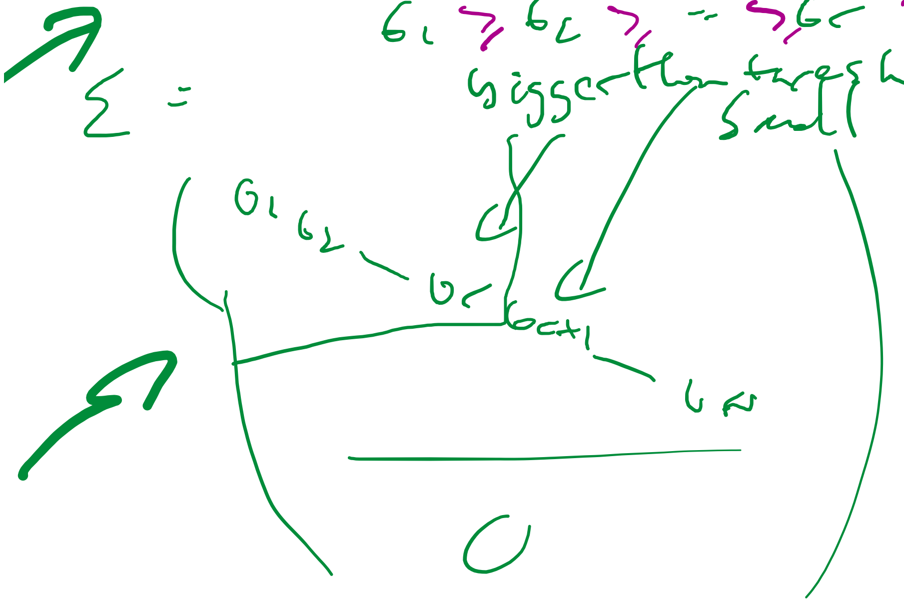
$$U = [U_1 | U_2 | \dots | U_r | \cancel{U_{r+1} | \dots | U_N}]$$

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq \sigma_{r+1} \geq \dots \geq \sigma_N$$

bigger than thresh

small

might be zero.



- choose r so that

$\sigma_r > \varepsilon$

$\sigma_{r+1} < \varepsilon$.

$$\Psi_r = [U_1 | \dots | U_r] \to U \in \mathbb{C}^{m \times r}$$

$$A = \underline{U}_r A_{DMD} \underline{U}_r^*$$

- projects A into POD basis to produce A_{DMD}

$$\boxed{A_{DMD} = \underline{U}^* A \underline{U}}$$

orthog. $\underline{U}_r \underline{U}_r^* = \underline{U}_r^* \underline{U}_r = I$

- $\underline{\Psi}_0 = \underline{U} \underset{\uparrow r}{\Sigma} V^*$

$$\underline{\Psi}_1 \overset{\underline{V}}{=} A \Psi_0 = A \underline{U} \Sigma V^* \cdot \underline{V}$$

$$\underline{\Psi}_1 \underline{V} \Sigma_+^{-1} = A \underline{U}$$

- $\Sigma_+^{-1} = \begin{pmatrix} 1/\sigma_1 & & & & & \\ & 1/\sigma_2 & & & & \\ & & \ddots & & & \\ & & & 1/\sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \; 0 \\ \hline & & 0 & & & \end{pmatrix}$ $\quad 1/\sigma_{r+1} \mapsto 0$

Ta Da!: $\boxed{A_{DMD} = \underline{U}^* \Psi_1 \cdot \underline{V} \Sigma^{-1} = \underline{U}^* A \underline{U}}$

$$A_{DMD} = \underline{\overline{U}}^{*} A \underline{\overline{U}} \qquad ; \quad \underline{\overline{\Phi}}_1 = A \underline{\overline{\Phi}}_0$$

$$= A \underline{\overline{U}} \Sigma \cancel{\underline{\overline{V}}^{T} V} \Sigma^{-1} \overset{*}{\underline{\overline{U}}}$$

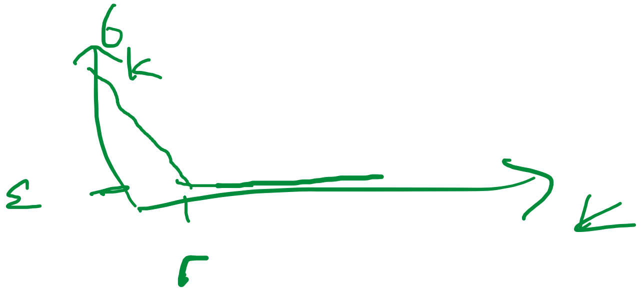
$$\left(\underline{\overline{\Phi}}_1 \underline{\overline{V}} \Sigma_1^{-1} \underline{\overline{U}}^{*} \right) = A$$

$$A_{DMD} = \underline{U^{*} \underline{\overline{\Phi}}_1 \underline{\overline{V}} \Sigma_1^{-1}} \cancel{\left(\underline{\overline{U}}^{*} \underline{\overline{U}} \right)}$$

$$= \underline{\overline{U}}^{*} \underline{\overline{\Phi}}_1 \underline{\overline{V}} \Sigma_1^{-1}$$

$A_{DMD_{r\times r}}$ is a low-rank, rank-r estimator of matrix $A_{M\times M}$ which is an estimator of Koopman operator

Works well to find low rank estimator when



A_{DMD} $r\times r$

$r \ll M$

A $M\times M$

- typically I don't really want to use to keep A or A_{DMD} (estimators of λ) eigenvalues & eigenvectors.
- Typical Behaviors in dynamic sense eigs of A_{DMD}

Using ADMD

∘ b_0 and amplitudes & DMD modes.

$$\underline{x_{t+1}} = \underline{A_{DMD}}\, \underline{x_t}$$

$r \times r$

r

$r \times 1$

Same x_t

φ_i

$m \times 1$

~~ADMD - φ_i doesn't~~ make sens.

$\underline{U}_r^{\#}$ $\qquad$ $\underline{U}_r^{*}$

$$\boxed{\varphi_t \approx \underline{U}_r\, x_t \;\Big|\; \Rightarrow x_t \approx \underline{U}_r^{*}\, \underline{\varphi}_t}$$

$\underline{U}_r \underline{U}_r = \underline{I}$

$r \times 1$ $\qquad$ $r \times m$ $\qquad$ $m \times 1$

$\underline{U}_r$

$m \times r$

$r \times 1$

- let $\{v_1, v_2, \ldots, v_r\}$ be eigenvectors of $\underbrace{A_{DMD}}_{r \times r}$ & $\{\lambda_1, \ldots, \lambda_r\}$ eigenvalues $\iff A_{DMD} v_i = \lambda_i v_i$

- But I want to work in r×1 space which sort artificial where I am doing easy compute — want is re-interpret back in M×1 space where my natural measurements are.

$$\underline{\Psi}_t \approx \overline{\underline{U}}_r \, x_t \leftarrow$$

not eigenvector of A_{DMD}

^expected to be

↑ not an eigenvector of A

- But then x_t may be a linear comb of eigenvectors of A_{DMD} , w_i ← $r \times r$

$A_{DMD} w_1 = \lambda_1 w_1$

$$x_0 = a_1 w_1 + a_2 w_2 + \dots + a_r w_r$$

$x_1 = A_{DMD} x_0$

$$x_1 = A_{DMD}(a_1 w_1 + a_2 w_2 + \dots + a_r w_r)$$

$$= a_1 \underline{A_{DMD} w_1} + a_2 A_{DMD} w_2 + \dots + a_r A_{DMD} w_r$$

$$= a_1 \lambda_1 w_1 + a_2 \lambda_2 w_2 + \dots + a_r \lambda_r w_r$$

change now to $x_t \approx \underline{U x_t}$

$$\boxed{x_t} = a_1 \lambda_1^t w_1 + a_2 \lambda_2^t w_2 + \dots + a_r \lambda_r^t w_r$$

$$= \sum^r a_i \lambda_i^t w_i$$

$\Psi_t \approx U x_t$

$\Psi_0 = U \Sigma v^*$

magnitudes

DMD modes

$$\boxed{\Psi_t = \sum_{i=1}^{r} \phi_i \lambda_i^t a_i}$$

λ_i

How DMD modes connect to POD modes.

$$\boxed{\Phi = [\phi_1\ \phi_2\ \dots\ \phi_r] = U_r W}$$

$$W = [w_1 | w_2 | \dots | w_r]_{r \times r}$$

eigenvectors of $\boxed{A_{DMD}}$

How to connect DMD modes to POD modes
W are eigs of A_DMD

ϕ_i are $m \times 1$ lifted w_i are $r \times 1$

lifted sys $\boxed{\phi_i = U_r w_i}$ $m \times r$ $i = 1, \dots, r$

$$\boxed{\phi_i = \underline{\bar{U}_r}\, W_i}$$

$i = 1 \dots r$

eigvect of A_{DMD} lifts by POD bases to be DMD modes.

- λ_i of A_{DMD} eigenvalues are taken as the eigenvalues of $\underline{A}$ estimators of what goes on $\mathcal{K}$.

$$\boxed{\Phi_0 = \bar{U}_r \underline{\bar{W}}\, \vec{a}} \quad \text{solve for } \vec{a}$$

$$\Phi_0 = \bar{U}_r x_0 = \bar{U}_r (a_1 W_1 + a_2 W_2 + \dots + a_r W_r)$$

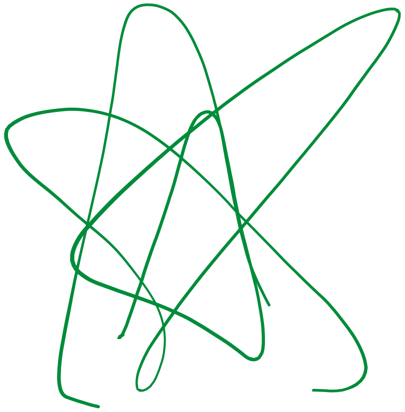
☆ $$\underline{\bar{\Phi}_0} = a_1 \bar{U}_r \underline{w}_1 + a_2 \underline{\bar{U}_r} \underline{w_2} + \dots + a_r \underline{\bar{U}_r}\, \underline{w_r}$$

$$\overset{\Psi_0}{\begin{bmatrix}\psi_0 & \psi_1 & \psi_2 & \cdots & \psi_{N-1}\end{bmatrix}} \approx \overset{\Phi}{\begin{bmatrix}\phi_0 & \phi_1 & \cdots & \phi_r\end{bmatrix}} \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_r \end{bmatrix} \overset{\lambda_i^0}{\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & \lambda_r & \lambda_r^2 & \cdots & \lambda_r^{N-1} \end{bmatrix}}$$

Φ matrix of DMD modes

amplitudes

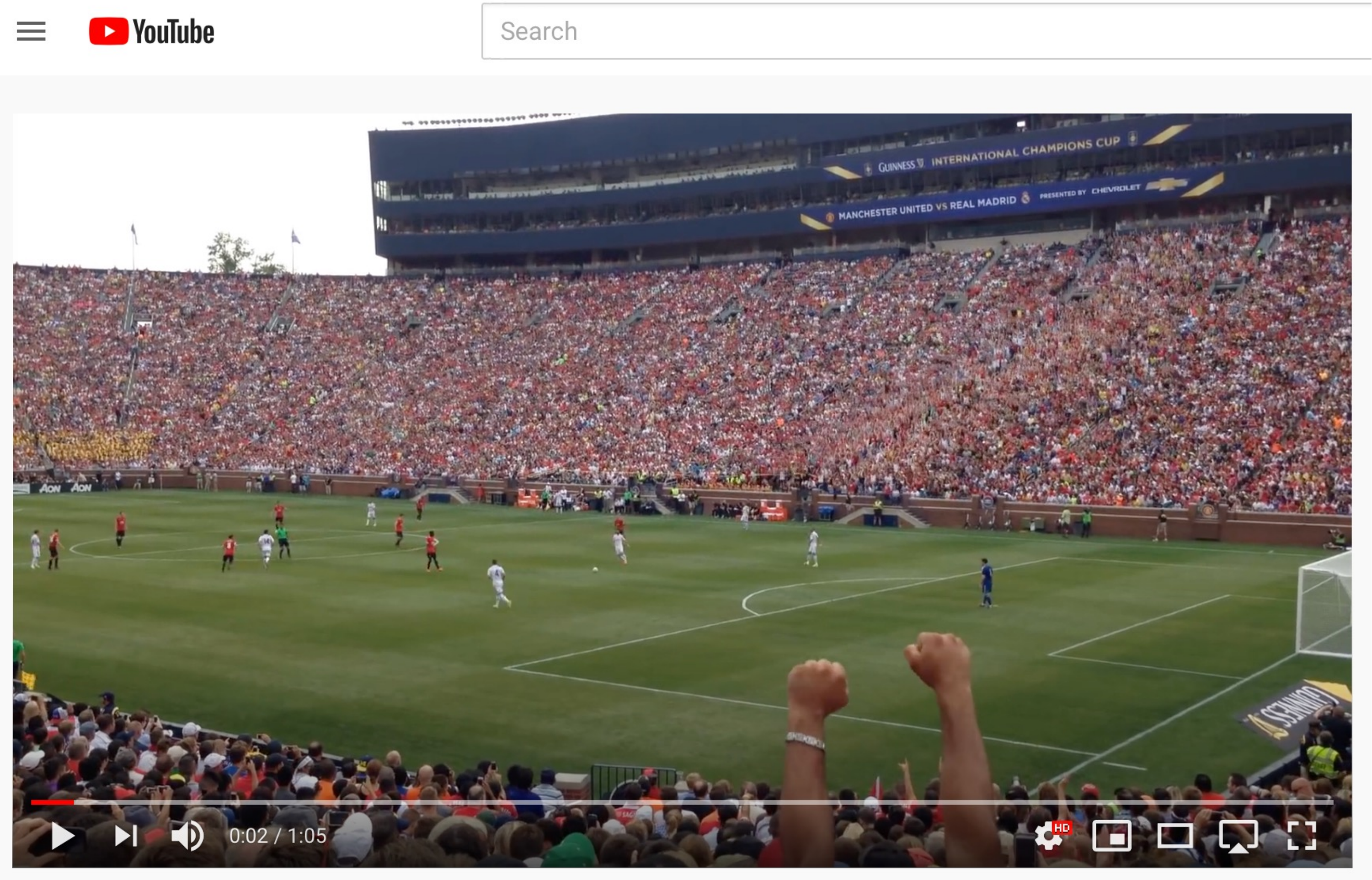
Vandermonde Matrix.



Wow

Its sort of a forecasting method.
Its mostly a descriptive metod.

What's moving? Thought experiment - The stadium wave.



Steps.

vector

- Experiment, collect data ψ_i at time snapshots t_i, then collect data matrices

  $\Psi_0 = [\psi_1, \psi_2, \ldots, \psi_{m-1}]$, $\Psi_1 = [\psi_2, \ldots, \psi_m]$

- Ansatz $\Psi_1 = A \Psi_0$, projected onto POD modes

  let $A_{DMD} = U^* A U$, U are POD modes of $\tilde{\Psi}_0 = U \Sigma V^*$

  let $\tilde{\Psi}_0 = \Psi_0 - \bar{\Psi}_0$

  $1 \times r$

  $r \times r$

- $$\boxed{A_{DMD} = U^* \Psi_1 V \Sigma^{-1}}$$

- λ_i, w_i are eVs of A_{DMD} (wrong size)

  $\phi_i = U_r w_i$

- $\Psi_0 = \Phi \vec{a}$ Vand ; and $\vec{a}$ solve linear $\psi_0 = \Phi \vec{a}$ vector ←

DMD for Strait of Gibraltar Data

Authors: Kanaththa Priyankara, Sathsara Dias, Sudam Surasinghe, with E. Bollt and M. Budišić (all at Clarkson University)

[Previous slides link]

[Larry's ppt]

Eigenvalue Spectrum

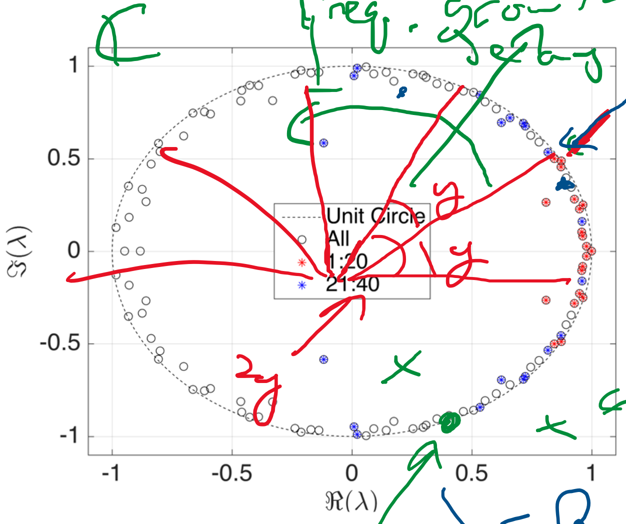
$|\lambda|^t \leftarrow |2\lambda|^t = |\lambda|^t|\omega|^t$

$(\lambda)^t = (e^{\omega})^t = e^{\omega t}$

$= e^{xt}(e^{iyt})$

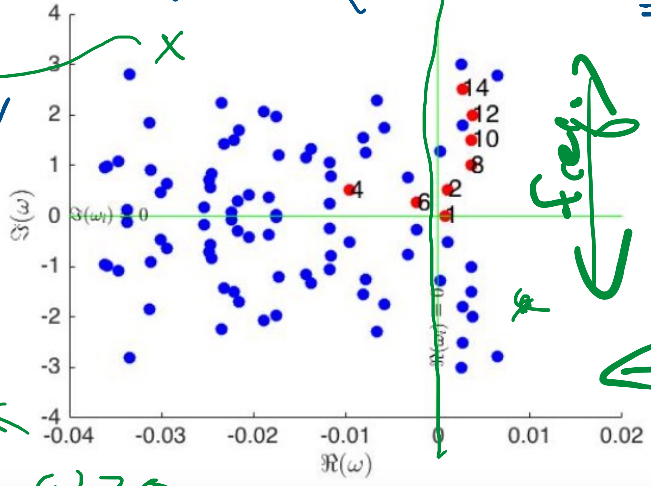
$e^{iyt} = \cos yt + i\sin$

freq. growth/decay



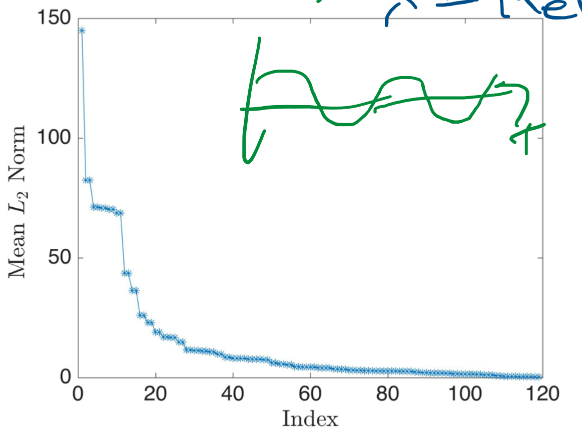
ϕx

$$\lambda_j (= e^{\omega_j})$$



conclude growth/freq.

$\lambda = Re(\lambda) + i\,Im(\lambda)$ $\quad \omega > 0$



$|\lambda| \leq 1$

$|\lambda|^t \xrightarrow{\text{if } |\lambda|<1} 0$

$\to \infty \quad > 1$

$x = Re(\omega)$ > 0, < 0, $= 0$

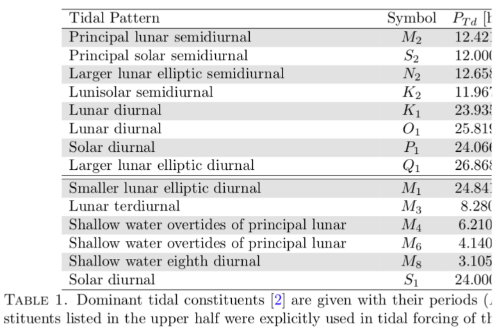
$|\lambda|^t = 1, \; |t| = 1$

$\omega = \overset{x}{Re\,\omega} + i\,\overset{y}{Im\,\omega}$

Modes

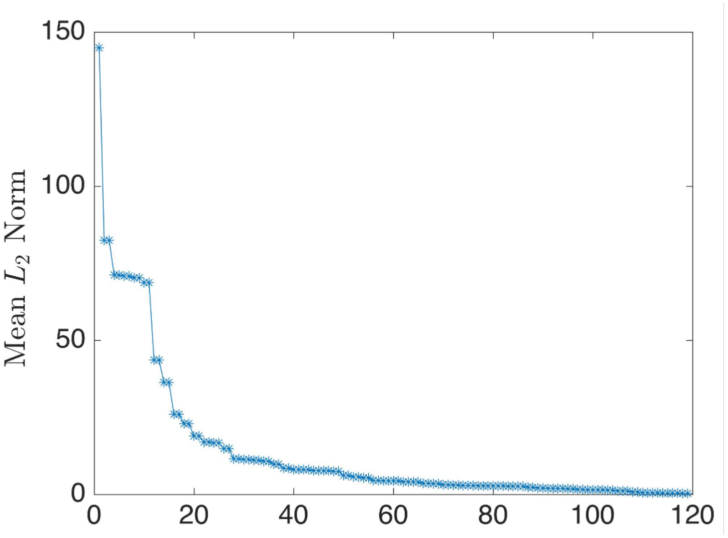
Full table

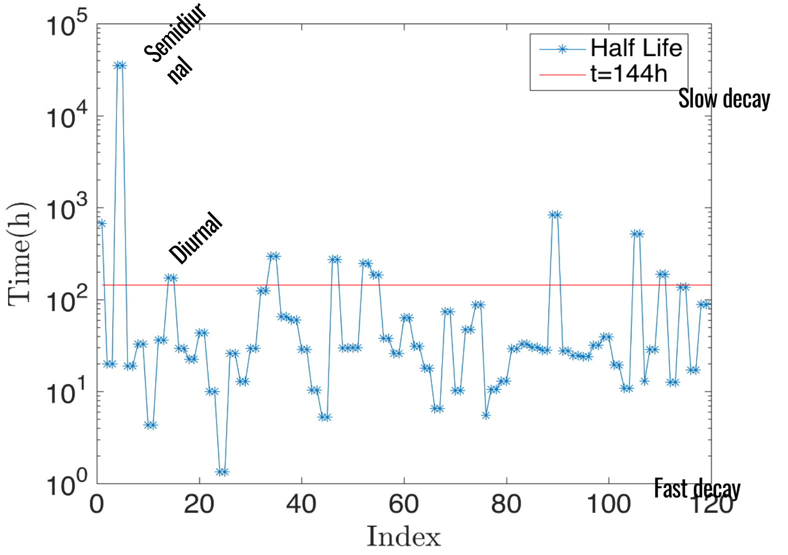
Index	HLT	Period	MeanL2Norm	|b|
1	668.10	Inf	144.84	155.63
3	19.91	-73.47	82.44	259.25
5	35116.86	-12.31	71.23	71.33
7	18.90	-57.93	70.85	228.63
9	32.88	228.92	70.24	172.08
11	4.33	19.87	68.73	463.55
13	36.37	-11.74	43.56	101.58
15	172.44	-24.65	36.32	47.04
17	29.34	26.61	25.94	67.23
19	22.42	21.34	22.97	68.08
21	43.32	-13.04	18.94	40.59
23	9.99	-7.48	16.99	75.44
25	1.34	3.55	16.71	202.43
27	25.96	37.70	14.85	40.91
29	12.87	4.02	11.52	45.04
31	29.35	-10.86	11.32	29.33
33	124.52	8.17	11.07	15.61
35	296.88	-6.24	10.78	12.60
37	64.91	8.37	9.81	17.51
39	59.96	4.06	8.45	15.61
41	28.69	-7.57	8.02	21.03
43	10.38	4.54	7.99	34.78
45	5.29	3.21	7.74	47.21
47	273.20	6.06	7.65	9.07
49	29.82	-17.42	7.42	19.08



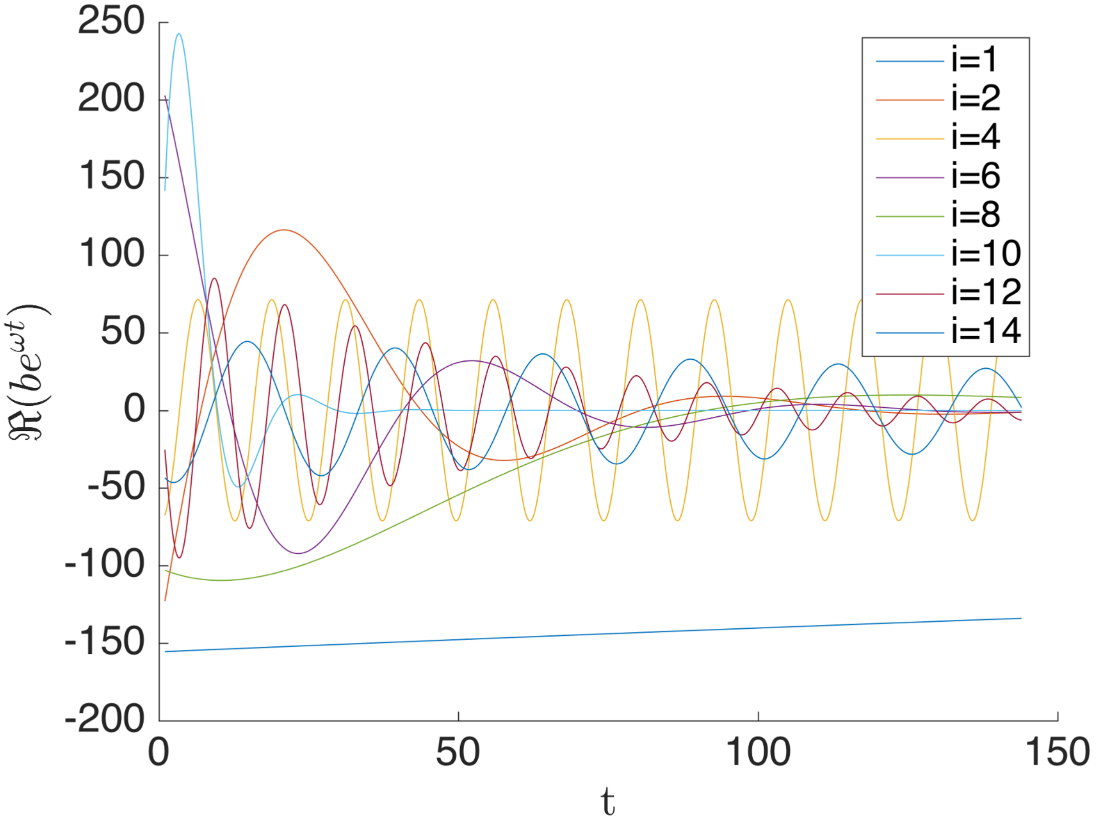
Tidal Pattern	Symbol	P_{Td} [
Principal lunar semidiurnal	M_2	12.42
Principal solar semidiurnal	S_2	12.00
Larger lunar elliptic semidiurnal	N_2	12.65
Lunisolar semidiurnal	K_2	11.96
Lunar diurnal	K_1	23.93
Lunar diurnal	O_1	25.81
Solar diurnal	P_1	24.06
Larger lunar elliptic diurnal	Q_1	26.86
Smaller lunar elliptic diurnal	M_1	24.84
Lunar terdiurnal	M_3	8.28
Shallow water overtides of principal lunar	M_4	6.210
Shallow water overtides of principal lunar	M_6	4.140
Shallow water eighth diurnal	M_8	3.105
Solar diurnal	S_1	24.00

TABLE 1. Dominant tidal constituents [2] are given with their periods (
stituents listed in the upper half were explicitly used in tidal forcing of th





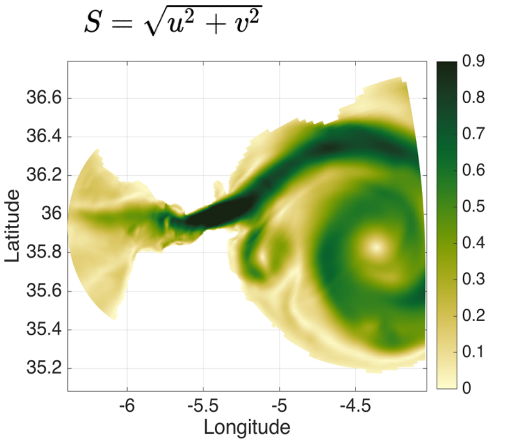
Time Dynamics: $\Re(be^{\omega t})$

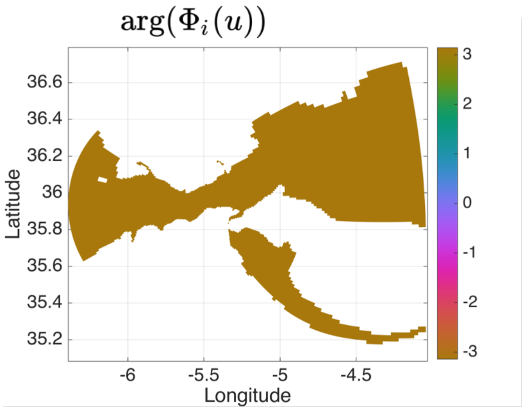


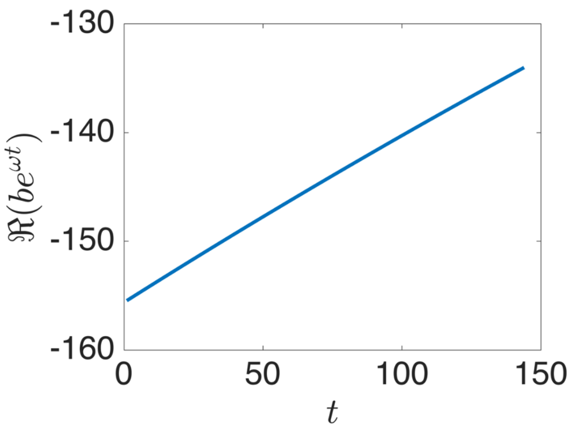
Tidal modes: low decay rate, concentrated, isolating tidal frequencies

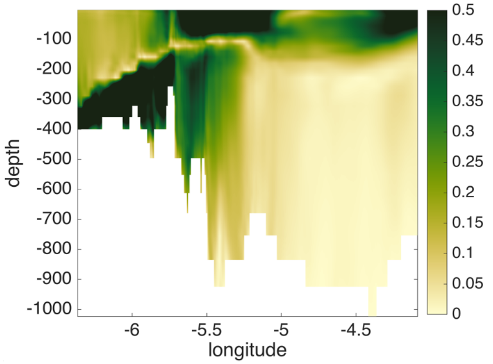
MODE 1: (Western Alboran Gyre+) HLT: 668.10 h PERIOD: inf

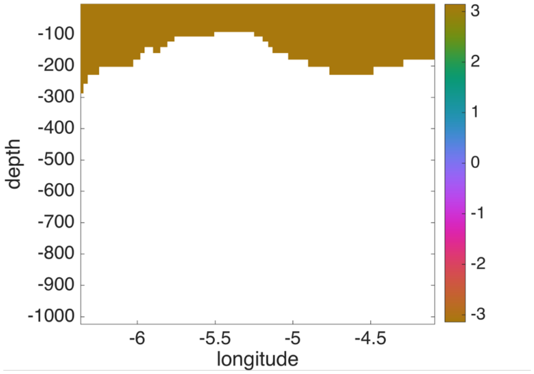
DMD mode 1 reveals the Western Alboran Gyre(WAG) and secondary gyre that sits between the Ceuta and the WAG. -











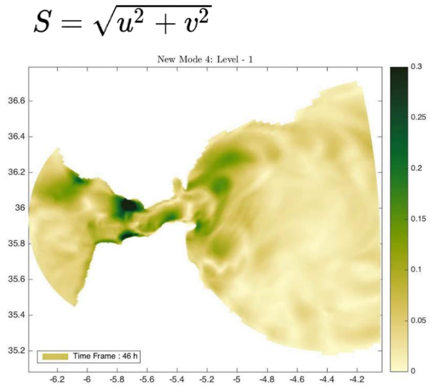
|b| : 155.63

Mean L2 : 144.84
(order by L2)

L2 of vertical velocity :
order by L2 vv

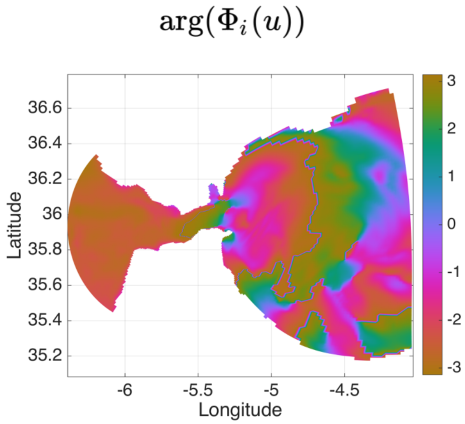
Mode 4: (semidiurnal)   HLT: 35116.86 h   PERIOD: 12.31 h

DMD mode 4(Previous Mode 2&4) reveals the semidiurnal Tidal mode with many qualitative features, including disorganized patches or swirls of high surface speed in the western Alboran Sea.

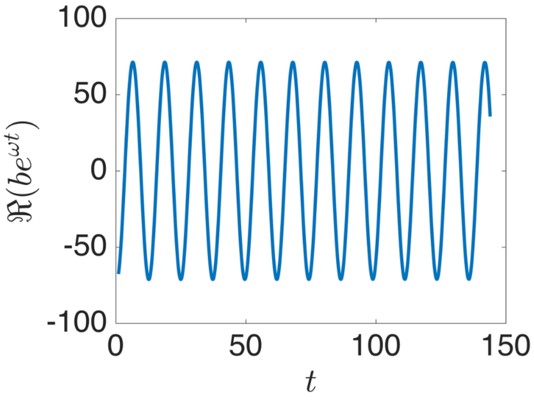


$S = \sqrt{u^2 + v^2}$

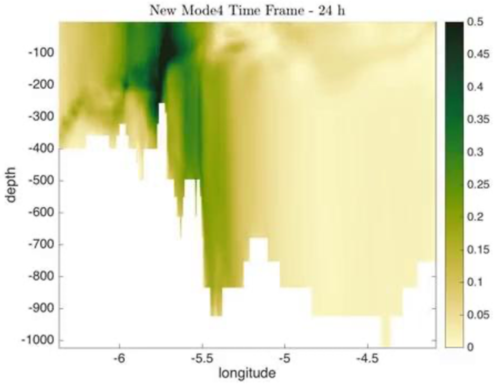
[redacted]

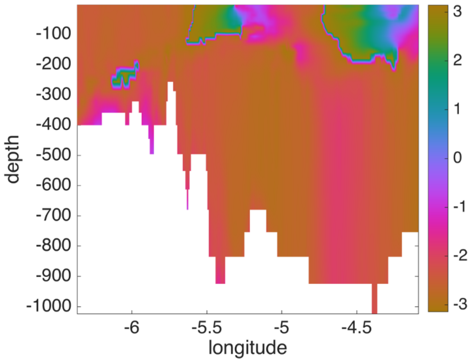


$\arg(\Phi_i(u))$



[redacted]





Mode 14: (diurnal) HLT: 172.44h PERIOD: 24.65h

DMD mode 14(Previously Known as mode 6) reveals the diurnal Tidal mode. One of the most striking features of this mode is the set of well defined bands of alternating surface velocity in the vicinity of the Atlantic Jet(AJ).