How can a nonlinear system (in 2D) be represented as a linear system?!!!

(a Linear System in Many (Infinite) Dimensions))

(By change of variables – A diffeomorphism)

Carleman Linearization – 1932 suggested by Henri Poincare'

Example NonLinear System in 2D

$$\begin{aligned} \dot{x}_1 &= x_1 + x_1 x_2 + x_2^2 \\ \dot{x}_2 &= x_1 + x_2. \end{aligned}$$

Substitute

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ y_3 &= x_1 x_2 \\ y_4 &= x_2^2. \end{aligned}$$

$$\dot{y}_1 = \dot{x_1} = \overset{\text{substitute } y_1}{\overset{\downarrow}{x_1}} + \overset{\text{substitute } y_3}{\overset{\downarrow}{(x_1 x_2)}} + \overset{\text{substitute } y_4}{\overset{\downarrow}{(x_2^2)}} = y_1 + y_3 + y_4.$$

$$\dot{y}_2 = \dot{x_2} = \overset{\text{substitute } y_1}{\overset{\downarrow}{x_1}} + \overset{\text{substitute } y_2}{\overset{\downarrow}{x_2}} = y_1 + y_2.$$

$$\begin{aligned} \dot{y_3} &= (x_1 \dot{x}_2) = \dot{x_1} x_2 + x_1 \dot{x_2} \text{ by product rule} \\ &= (x_1 + x_1 x_2 + x_2^2) x_2 + x_1 (x_1 + x_2) \text{ by substitution of equations for } \dot{x_1} \text{ and } \dot{x_2}. \\ &= \overset{\text{substitute } y_3}{\overset{\downarrow}{2(x_1 x_2)}} + \overset{\text{new declare } y_5}{\overset{\downarrow}{(x_1^2)}} + \overset{\text{new declare } y_6}{\overset{\downarrow}{(x_1 x_2^2)}} + \overset{\text{new declare } y_7}{\overset{\downarrow}{(x_2^3)}} \\ &= 2y_3 + y_5 + y_6 + y_7 \\ \dot{y_4} &= (\dot{x_2^2}) = 2\dot{x_2} x_2 \text{ by chain rule} \\ &= 2x_2 (x_1 + x_2) \text{ by substitution of equations for } \dot{x_2} \\ &= \overset{\text{substitute } y_3}{\overset{\downarrow}{2(x_1 x_2)}} + \overset{\text{substitute } y_4}{\overset{\downarrow}{2(x_2^2)}} \\ &= 2y_3 + 2y_4. \end{aligned}$$

$$
\begin{aligned}
\dot{y_5} &= \dot{(x_1^2)} = 2\dot{x_1}x_1 = 2(x_1 + x_1x_2 + x_2^2)x_1 \\
&\qquad \text{substitute } y_5 \qquad \text{new declare } y_8 \qquad \text{substitute } y_6 \\
&= \underset{\text{substitute } y_5}{2(x_1^2)} + \underset{\text{new declare } y_8}{2(x_1^2x_2)} + \underset{\text{substitute } y_6}{2(x_1x_2^2)} \\
&= 2y_5 + +2y_6 + 2y_8 \\
\dot{y_6} &= \dot{(x_1x_2^2)} = \dot{x_1}x_2^2 + x_1(2\dot{x_2}x_2) = (x_1 + x_1x_2 + x_2^2)x_2^2 + 2x_1x_2(x_1 + x_2) \\
&= \underset{\text{substitute } y_3}{(x_1x_2)} + \underset{\text{substitute } y_6}{3(x_1x_2^2)} + \underset{\text{substitute } y_7}{(x_2^3)} + \underset{\text{substitute } y_8}{2(x_1^2x_2)} \\
&= y_3 + 3y_6 + y_7 + 2y_8.
\end{aligned}
$$

$$
\begin{aligned}
\dot{y_7} &= \dot{(x_2^3)} = 3x_2^2\dot{x_2} = 3x_2(x_1 + x_2) \\
&= \underset{\text{substitute } y_6}{3(x_1x_2^2)} + \underset{\text{substitute } y_7}{3(x_2^3)} \\
&= 3y_6 + 3y_7 \\
\dot{y_8} &= \dot{(x_1^2x_2)} = 2\dot{x_1}x_1x_2 + x_1^2\dot{x_2} \\
&= \underset{\text{substitute } y_8}{3(x_1^2x_2)} + \underset{\text{new declare } y_9}{2(x_1^2x_2^2)} + \underset{\text{new declare } y_{10}}{2(x_1x_2^3)} + \underset{\text{substitute } y_7}{x_1^3} \\
&= y_7 + 3y_8 + 2y_9 + 2y_{10}.
\end{aligned}
$$

Interpret as a change of variable.
h:R^2->R^N

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ \vdots \end{pmatrix} = h\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \\ x_1 x_2 \\ x_2^2 \\ x_1^2 \\ x_1 x_2^2 \\ x_2^3 \\ x_1^2 x_2 \\ x_1^2 x_2^2 \\ x_1 x_2^3 \\ \vdots \end{pmatrix}$$

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The linear equations in high dimensions

$$\underset{\dot{y}}{\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \\ \dot{y}_6 \\ \dot{y}_7 \\ \dot{y}_8 \\ \vdots \end{pmatrix}} = \underset{A}{\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & \dots \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 & 2 & \dots \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & \dots \\ \vdots & \vdots & & & & & & & \ddots \end{pmatrix}} \underset{y}{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ \vdots \end{pmatrix}}$$

EDMD is DMD in terms of basis functions

why:

$$\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \end{matrix} = \begin{matrix} x_1 + x_1 x_2 + x_2^2 \\ x_1 + x_2 \end{matrix}$$

linear? $\dot{x} = Ax$?!

No.

How about

$y_1(t) = x_1(t)$

$y_2(t) = x_2(t)$

$y_3(t) = x_1(t-1)$

$y_4(t) = x_2(t-1)$

$y_5(t) = x_1(t-2)$

$\vdots$

$y_N(t) = x_2(t - \frac{N}{2} +)$

or spatiotemporal

KS equations y_1:

Assume $y_1(t+1) = a_0 + a_1 y_1(t) + \dots + a_N y_N(t)$
$+ a_{N+1} y_1^2(t) + \dots$ etc..

Lets keep it simple

$y_1 = x_1,\ y_2 = x_2,\ y_3 = x_1 x_2,\ y_4 = x_2^2,\ y_5 = x_1^2, \dots$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \end{pmatrix} = h\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_1 x_2 \\ x_2^2 \\ x_1^2 \\ x_1 x_2^2 \\ x_2^3 \\ \vdots \end{pmatrix}$$

$\psi_1(x) \to x_1$, $\psi_2(x) \to x_2$, $\psi_3(x) \to x_1 x_2$

Linear terms — quadratic terms — cubic terms $\vdots$

$h: \mathbb{R}^2 \to \mathbb{R}^N$ diffeo

Anzatz

EDMD from $X_{2 \times m-1}$ form $\Psi_{N \times m-1}$ then do DMD

$$\Psi' = A\Psi$$

On DMDc

why:  ← u(t) vs.

$\ddot{x} + k\sin x = 0$ $\xrightarrow[\dot y = \ddot x]{y = \dot x}$ $\dot y = -k\sin x$, $\dot x = y$

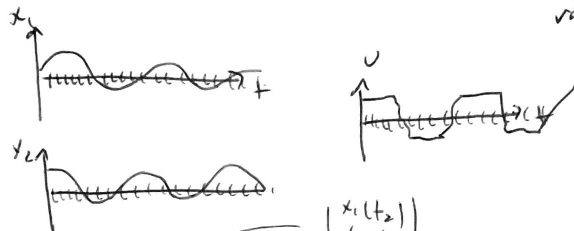
$\ddot{x} + k\sin x = au$ → $\dot x = y$, $\dot y = -k\sin x + a$

when angle small, $x \ll 1 \Rightarrow \sin x \approx x$ → Harmonic Oscillator

$$\begin{pmatrix} \dot x_1 \\ \dot x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

← u ← "Exogenous variables"

the setup: $\dot x = Ax + Bu$



$$\underline{X} = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_{m-1} \\ | & | & & | \end{bmatrix}; \quad \underline{X}' = \begin{bmatrix} | & | & & | \\ x_2 & x_3 & \dots & x_m \\ | & | & & | \end{bmatrix}$$

(x_1 ↔ $\begin{pmatrix} x_1(t_1) \\ x_2(t_1) \end{pmatrix}$, x_2 ↔ $\begin{pmatrix} x_1(t_2) \\ x_2(t_2) \end{pmatrix}$)

$$\underline{U} = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_{m-1} \\ | & | & & | \end{bmatrix}$$

(u_1 ↔ $\begin{pmatrix} u_1(t_1) \\ u_2(t_1) \end{pmatrix}$)

let $z_i = \begin{pmatrix} x_i \\ u_i \end{pmatrix} \Rightarrow \underline{Z} = \begin{bmatrix} \underline{X} \\ \underline{U} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_{m-1} \\ | & | & & | \\ u_1 & u_2 & & u_{m-1} \\ | & | & & | \end{bmatrix}$

Anzatz, assume (linear) $\underline{X}' = \underbrace{[A \;\; B]}_{C} \underbrace{\begin{bmatrix} \underline{X} \\ \underline{U} \end{bmatrix}}_{Z} = CZ$

the solution ⇒ $C = [A \;\; B] := \underline{X}' \underline{Z}^+$ (ls estimator) solves $\min \|\underline{X}' - C\underline{Z}\|_F$