

One-Dimensional Continuous-Time Quantum Walks

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We survey the equations of continuous-time quantum walks on simple one-dimensional lattices, which include the finite and infinite lines and the finite cycle, and compare them with the classical continuous-time Markov chains. The focus of our expository article is on analyzing these processes using the Laplace transform on the stochastic recurrences. The resulting time evolution equations, classical vs. quantum, are strikingly similar in form, although dissimilar in behavior. We also provide comparisons with analyses performed using spectral methods.

KEY WORDS: Quantum walks; continuous time; Laplace transform.

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1. INTRODUCTION

The theory of Markov chains on countable structures is an important area in Mathematics and Physics. A quantum analog of continuous-time Markov chains on the infinite line is well studied in Physics (for example, it can be found in Ref. 12, Chapters 13 and 16). More recently, it was studied by Aharonov *et al.*⁽²⁾ and by Farhi and Gutmann.⁽¹¹⁾ The latter work placed the problem in the context of quantum algorithms for search problems on graphs. Here, the symmetric stochastic matrix of the graph is viewed as a Hamiltonian of the quantum process. Using Schrödinger's equation with this Hamiltonian, we obtain a quantum walk on the underlying graph, instead of a classical random walk.

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Recent works on continuous-time quantum walks on finite graphs include the analyses of mixing and hitting times on the n -cube,^(16,14) of mixing times on circulant graphs and Cayley graphs of the symmetric group,^(6,13) and of hitting times on path-like graphs.^(8,9) Most of these are structural results based on spectral analysis of the underlying graphs, such as the n -cube, circulant and Cayley graphs, and (weighted) paths. For example, Moore and Russell⁽¹⁶⁾ proved that the mixing time of a quantum walk on the n -cube is asymptotically faster than a classical random walk; Kempe⁽¹⁴⁾ proved that the hitting time for vertices on opposite ends of the n -cube is exponentially faster than in a classical random walk. Ahmadi *et al.*⁽⁶⁾ and Gerhardt and Watrous⁽¹³⁾ proved that circulants and the Cayley graph of the symmetric group lack the uniform mixing property found in classical random walks.

A recent work of Childs *et al.*⁽⁹⁾ gave intriguing evidence that continuous-time quantum walk is a powerful method for designing new quantum algorithms. They analyzed diffusion processes on one-dimensional structures (finite path and infinite line) using spectral methods. Another work by Childs and Goldstone⁽¹⁰⁾ explored the application of continuous-time quantum walks to perform Grover search on spatial lattices.

There is an alternate theory of *discrete* quantum walks on graphs, which we will not discuss here. This alternate model was studied in Aharonov *et al.*⁽⁴⁾ and Ambainis *et al.*,⁽⁵⁾ but had appeared earlier in work by Meyer.⁽¹⁵⁾ The work by Ambainis *et al.*⁽⁵⁾ had also focused on one-dimensional lattices. Recently, Ambainis⁽¹⁾ developed an optimal (discrete) quantum walk algorithm for the fundamental problem of Element Distinctness. This offers another idea for developing quantum algorithms.

We survey and (re)derive equations for the continuous-time classical and quantum walks on one-dimensional lattices using the Laplace transform that works directly with the recurrences. The Laplace transform is a well-known tool in stochastic processes (see Ref. 7) and it might offer a useful alternative to the Fourier transform in certain settings.

1.1. Stochastic Walks on Graphs

Let $G = (V, E)$ be a simple (no self-loops), countable, undirected graph with adjacency matrix A . Let D be a diagonal matrix whose j th entry is the degree of the j th vertex of G . The Laplacian of G is defined as $H = A - D$. Suppose that $P(t)$ is a time-dependent probability distribution of a stochastic (particle) process on G . The classical evolution of the continuous-time walk is given by the Kolmogorov equation

$$P'(t) = HP(t). \quad (1)$$

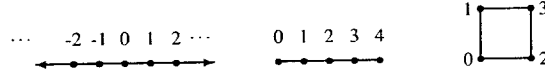


Fig. 1. Examples of some one-dimensional lattices. From left to right: \mathbb{Z} , \mathbb{P}_4 , \mathbb{Z}_4 .

Graph	Classical walk	Quantum walk
	$P_j(t) =$ probability on vertex j at time t	$\psi_j(t) =$ amplitude on vertex j at time t
\mathbb{Z}	$e^{-t}I_{ j }(t)$	$(-i)^{ j }J_{ j }(t)$
\mathbb{Z}_N	$\sum_{\alpha \equiv \pm j \pmod{N}} e^{-t}I_{\alpha}(t)$	$\sum_{\alpha \equiv \pm j \pmod{N}} (-i)^{\alpha}J_{\alpha}(t)$
\mathbb{P}_N	$\sum_{\alpha \equiv \pm j \pmod{2N}} e^{-t}I_{\alpha}(t)$	$\sum_{\alpha \equiv \pm j \pmod{2N}} (-i)^{\alpha}J_{\alpha}(t)$

Fig. 2. The equations of the continuous-time classical vs. quantum walks on the infinite line, finite cycle, and the finite line, assuming the particle starts at position 0.

The solution of this equation, modulo some conditions, is $P(t) = e^{tH}P(0)$, which can be solved by diagonalizing the symmetric matrix H . This *spectral* approach requires full knowledge of the spectrum of H .

A quantum analog of this classical walk uses the Schrödinger equation in place of the Kolmogorov equation. Let $\psi : V(G) \rightarrow \mathbb{C}$ be the time-independent amplitude of the quantum process on G . Then, the wave evolution is

$$i\hbar \frac{d}{dt} \psi(t) = H \psi(t). \tag{2}$$

Assuming $\hbar = 1$ for simplicity, the solution of this equation is $\psi(t) = e^{-itH} \psi(0)$, which, again, is solvable via spectral techniques. The classical behavior of this quantum process is given by the probability distribution $P(t)$ whose j th entry is $P_j(t) = |\psi_j(t)|^2$, where $\psi_j(t) = \langle j | \psi(t) \rangle$. The *average probability* of vertex j is defined as $\bar{P}(j) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T P_j(t) dt$ (see Ref. 4) (Fig. 1).

The table in Fig. 2 shows the known equations for continuous-time stochastic walks on the infinite (integer) line $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the finite cycle $\mathbb{Z}_N = \{0, \dots, N - 1\}$ on N vertices, and the finite path $\mathbb{P}_N = \{0, \dots, N\}$ on $N + 1$ vertices, in terms of the two kinds of Bessel functions $I(\cdot)$ and $J(\cdot)$. We assume here that the particle is initially at 0. The plots in Figs. 3 and 4 show the dissimilar behavior of the classical versus quantum walks.

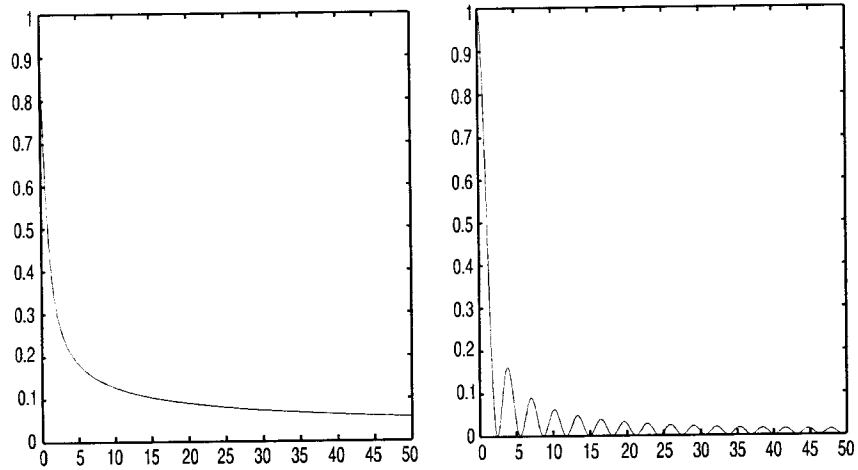


Fig. 3. Stochastic walks on the infinite line \mathbb{Z} : (a) plot of $P_0(t)$ in the continuous-time random walks for $t \in [0, 50]$. (b) plot of $|\psi_0(t)|^2$ in a continuous-time quantum walk for $t \in [0, 50]$. Both processes exhibit exponential decay, but with the quantum walk showing an oscillatory behavior.

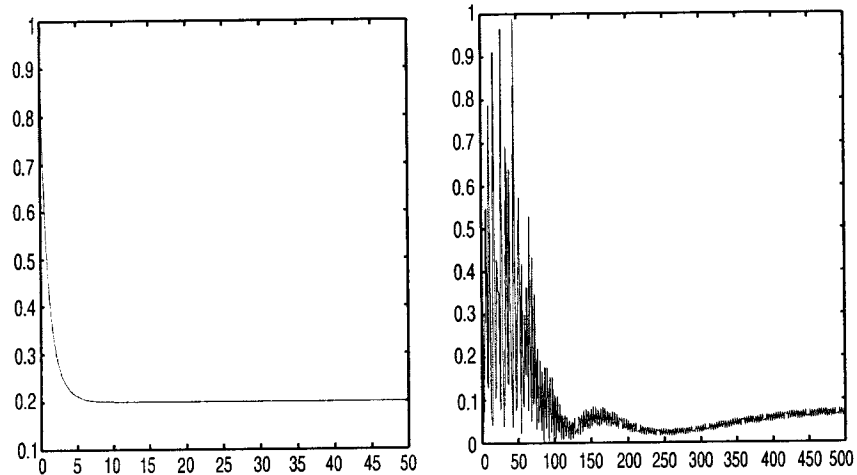


Fig. 4. Stochastic walks on the finite cycle \mathbb{Z}_5 , each approximated using 21 terms: (a) plot of $P_0(t)$ in the continuous-time random walks for $t \in [0, 50]$. (b) plot of $|\psi_0(t)|^2$ in the continuous-time quantum walk for $t \in [0, 500]$. The classical walk settles quickly to $1/5$, while the quantum walk exhibit a short-term chaotic behavior and a long-term oscillatory behavior below 0.1 .

1.2. Laplace Transform

The Laplace transform of a time-dependent function $P(t)$, denoted $\hat{P}(s) = \mathcal{L}\{P(t)\}$, is defined as

$$\mathcal{L}\{P(t)\} = \int_0^{\infty} e^{-st} P(t) dt. \quad (3)$$

The only basic properties of the Laplace transform which we will need are (see Ref. 3):

- Linearity: $\mathcal{L}\{aP(t) + bQ(t)\} = a\hat{P}(s) + b\hat{Q}(s)$
- Derivative: $\mathcal{L}\{P'(t)\} = s\hat{P}(s) - P(0)$
- Shifting: $\mathcal{L}\{e^{at}P(t)\} = \hat{P}(s - a)$

The relevant inverse Laplace transform involving the Bessel functions are (for $\nu > -1$):

$$\hat{P}(s) = \frac{(s - \sqrt{s^2 - a^2})^\nu}{\sqrt{s^2 - a^2}} \iff P(t) = a^\nu I_\nu(at), \quad (\text{Eq. (29.3.59) in Ref. 3}) \quad (4)$$

$$\hat{P}(s) = \frac{(\sqrt{s^2 + a^2} - s)^\nu}{\sqrt{s^2 + a^2}} \iff P(t) = a^\nu J_\nu(at). \quad (\text{Eq. (29.3.56) in Ref. 3}) \quad (5)$$

2. THE INFINITE LINE

2.1. Classical Process

The Kolmogorov equation for the infinite line is

$$P_j'(t) = \frac{1}{2}P_{j-1}(t) - P_j(t) + \frac{1}{2}P_{j+1}(t), \quad (6)$$

with initial value $P_j(0) = \delta_{0,j}$. The Laplace transform of Eq. (6) is

$$\hat{P}_{j+1}(s) - 2(s+1)\hat{P}_j(s) + \hat{P}_{j-1}(s) = -P_j(0). \quad (7)$$

The solution of $q^2 - 2(s+1)q + 1$ is $q_{\pm} = (s+1) \pm \sqrt{(s+1)^2 - 1}$. A natural guess of the solution is

$$\hat{P}_j(s) = \begin{cases} Aq_+^j & \text{if } j < 0 \\ Aq_-^j & \text{if } j > 0. \end{cases} \quad (8)$$

When $j=0$, we get $A=(1+s-q_-)^{-1}$. Thus, for $j \in \mathbb{Z}$,

$$\hat{P}_j(s) = \frac{q_-^{|j|}}{(1+s-q_-)} = \frac{[(s+1) - \sqrt{(s+1)^2 - 1}]^{|j|}}{\sqrt{(s+1)^2 - 1}}. \quad (9)$$

Using the inverse Laplace transform Eq. (4), after shifting $S=s+1$, we get

$$P_j(t) = e^{-t} I_{|j|}(t). \quad (10)$$

This is a probability function, since $e^{t/2(z+1/z)} = \sum_{k=-\infty}^{\infty} z^k I_k(t)$, if $z \neq 0$ (see Eq. (9.6.33) in Ref. 3).

2.2. Quantum Process

The Schrödinger equation for the infinite line is

$$i\psi'_j(t) = \frac{1}{2}\psi_{j-1}(t) + \frac{1}{2}\psi_{j+1}(t). \quad (11)$$

The Laplace transform of Eq. (11) is

$$\hat{\psi}_{j+1}(s) - 2i(s\hat{\psi}_j(s) - \psi_j(0)) + \hat{\psi}_{j-1}(s) = 0. \quad (12)$$

The solutions of $q^2 - 2isq + 1$ are $q_{\pm} = i(s \pm \sqrt{s^2 + 1})$, where $q_+q_- = 1$. A guess for the solution is

$$\hat{\psi}_j(s) = \begin{cases} Aq_+^j & \text{if } j \leq 0 \\ Aq_-^j & \text{if } j > 0. \end{cases} \quad (13)$$

When $j=0$, we get $A=(s+iq_-)^{-1}$. Thus,

$$\hat{\psi}_j(s) = \frac{q_-^{|j|}}{(s+iq_-)} = (-i)^{|j|} \frac{(\sqrt{s^2+1}-s)^{|j|}}{\sqrt{s^2+1}}. \quad (14)$$

The inverse Laplace transform Eq. (5) yields, for $j \in \mathbb{Z}$,

$$\psi_j(t) = (-i)^{|j|} J_{|j|}(t). \quad (15)$$

This is a probability function, since $1 = J_0^2(z) + 2\sum_{k=1}^{\infty} J_k^2(z)$ (see Eq. (9.1.76) in Ref. 3).

2.3. Spectral Analysis

Let H be a Hamiltonian defined as $\langle j|H|k\rangle = \frac{1}{2}$ if $j = k \pm 1$, and 0 otherwise. For each $p \in [-\pi, \pi]$, define $|p\rangle$ so that

$$\langle j|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipj}. \quad (16)$$

The eigenvalue equation $H|p\rangle = \lambda_p|p\rangle$ has the solution $\lambda_p = \cos(p)$. Thus, the amplitude of position j when the particle starts at position 0 is

$$\begin{aligned} \langle j|e^{-iHt}|0\rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ipj} e^{-it \cos(p)} dp \\ &= (-i)^j J_j(t). \quad (\text{see Eq. (9.1.21) in Ref. 3}) \end{aligned} \quad (17)$$

Childs *et al.*⁽⁹⁾ gave a more generalized analysis along these lines.

3. THE FINITE CYCLE

3.1. Classical Process

If A is the adjacency matrix of the finite cycle, let $H = \frac{1}{2}A - I$ be its Laplacian matrix. The Kolmogorov equation for the finite cycle is

$$P'_j(t) = \frac{1}{2}P_{j-1}(t) - P_j(t) + \frac{1}{2}P_{j+1}(t). \quad (18)$$

Applying the Laplace transform to Eq. (18), we get

$$(s+1)\hat{P}_j(s) - P_j(0) = \frac{1}{2}\hat{P}_{j-1}(s) + \frac{1}{2}\hat{P}_{j+1}(s). \quad (19)$$

For convenience, define the extra condition $\hat{P}_{-1}(s) = \hat{P}_{N-1}(s) + 2$, so that $\hat{P}_{j+1}(s) - 2(s+1)\hat{P}_j(s) + \hat{P}_{j-1}(s) = 0$ holds for $j \in \mathbb{Z}_N$. The cycle condition is $\hat{P}_N(s) = \hat{P}_0(s)$. We guess the solution to be

$$\hat{P}_j(s) = Aq_+^j + Bq_-^j, \quad (20)$$

where q_{\pm} is the solution to $x^2 - 2(s+1)x + 1 = 0$, i.e., $q_{\pm} = (s+1) \pm \sqrt{(s+1)^2 - 1}$, with $q_+q_- = 1$. Using the cycle condition, we get

$$Aq_+^N + Bq_-^N = A + B \implies A(q_+^N - 1) = B(1 - q_-^N) \implies B = Aq_+^N. \quad (21)$$

Using the extra condition and Eq. (21), we get $A = 2((q_+ - q_-)(q_+^N - 1))^{-1}$. Thus, for $j \in \mathbb{Z}_N$,

$$\begin{aligned} \hat{P}_j(s) &= Aq_+^j + Bq_-^j = A(q_+^j + q_+^{N-j}) \\ &= \frac{2}{(q_+ - q_-)} \frac{(q_+^j + q_+^{N-j})}{(1 - q_+^N)} = \frac{2}{(q_+ - q_-)} \sum_{k=0}^{\infty} (q_+^{kN+j} + q_+^{(k+1)N-j}) \\ &= \sum_{k=0}^{\infty} \left[\frac{((s+1) - \sqrt{(s+1)^2 - 1})^{kN+j}}{\sqrt{(s+1)^2 - 1}} + \frac{((s+1) - \sqrt{(s+1)^2 - 1})^{(k+1)N-j}}{\sqrt{(s+1)^2 - 1}} \right]. \end{aligned}$$

The inverse Laplace transform Eq. (4), after shifting, yields, for $j \in \mathbb{Z}_N$,

$$P_j(t) = \sum_{k=0}^{\infty} e^{-t} [I_{kN+j}(t) + I_{(k+1)N-j}(t)] = \sum_{\alpha \equiv \pm j \pmod{N}} e^{-t} I_{\alpha}(t). \quad (22)$$

3.2. Quantum Process

Since the finite cycle is a regular graph, instead of the Laplacian, we use the adjacency matrix directly. In a continuous-time quantum walk, this simply introduces an irrelevant phase factor in the final expression. The Schrödinger equation, in this case, is

$$i\psi'_j(t) = \frac{1}{2}\psi_{j-1}(t) + \frac{1}{2}\psi_{j+1}(t). \quad (23)$$

The Laplace transform of Eq. (23) is

$$\hat{\psi}_{j+1}(s) - 2i(s\hat{\psi}_j(s) - \psi_j(0)) + \hat{\psi}_{j-1}(s) = 0. \quad (24)$$

The cycle boundary condition is $\hat{\psi}_N(s) = \hat{\psi}_0(s)$. For convenience, define

$$\hat{\psi}_{-1}(s) = \hat{\psi}_{N-1}(s) = 2i. \quad (25)$$

The solutions of $q^2 - 2isq + 1$ are $q_{\pm} = i(s \pm \sqrt{s^2 + 1})$, with $q_+q_- = 1$. A solution guess, for $j \in \mathbb{Z}_N$, is

$$\hat{\psi}_j(s) = Aq_+^j + Bq_-^j. \quad (26)$$

The cycle boundary condition yields $B = Aq_+^N$. By Eq. (25), we get $A = 2i((q_+ - q_-)(q_+^N - 1))^{-1}$. Thus, for $j \in \mathbb{Z}_N$,

$$\begin{aligned} \hat{\psi}_j(s) &= Aq_+^j + Bq_-^j = A(q_+^j + q_+^{N-j}) \\ &= \frac{2i}{(q_+ - q_-)} \frac{(q_-^j + q_-^{N-j})}{(1 - q_-^N)} = \frac{2i}{(q_+ - q_-)} \sum_{k=0}^{\infty} (q_-^{kN+j} + q_-^{(k+1)N-j}) \\ &= \sum_{k=0}^{\infty} \left[\frac{((-i)(\sqrt{s^2+1}-s))^{kN+j}}{\sqrt{s^2+1}} + \frac{((-i)(\sqrt{s^2+1}-s)^{(k+1)N-j}}{\sqrt{s^2+1}} \right]. \end{aligned}$$

The inverse Laplace transform Eq. (5) gives, for $j \in \mathbb{Z}_N$,

$$\begin{aligned} \psi_j(t) &= \sum_{k=0}^{\infty} \left[(-i)^{kN+j} J_{kN+j}(t) + (-i)^{(k+1)N-j} J_{(k+1)N-j}(t) \right] \\ &= \sum_{\alpha \equiv \pm j \pmod{N}} (-i)^\alpha J_\alpha(t). \end{aligned} \tag{27}$$

3.3. Spectral Analysis

The normalized adjacency matrix H of \mathbb{Z}_N is the circulant matrix

$$H = \begin{pmatrix} 0 & 1/2 & 0 & \dots & 0 & 1/2 \\ 1/2 & 0 & 1/2 & \dots & 0 & 0 \\ 0 & 1/2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1/2 & 0 & 0 & \dots & 1/2 & 0 \end{pmatrix}. \tag{28}$$

It is well known that all $N \times N$ circulant matrices are unitarily diagonalized by the Fourier matrix $F = \frac{1}{\sqrt{N}}V(\omega_N)$, where $\omega_N = e^{2\pi i/N}$ and $V(\omega_N)$ is the Vandermonde matrix defined as $V(\omega_N)[j, k] = \omega_N^{jk}$, for $j, k \in \{0, 1, \dots, N-1\}$. The eigenvalues of H are $\lambda_j = \frac{1}{2}(\omega_N^j + \omega_N^{j(N-1)}) = \cos(2\pi j/N)$, for $j = 0, 1, \dots, N-1$. Thus, the wave amplitude at vertex j at time t is

$$\psi_j(t) = \frac{1}{N} \sum_{k=0}^{N-1} e^{-it \cos(2\pi k/N)} \omega_N^{jk}. \tag{29}$$

From earlier analysis, we get the following Bessel equation

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{-it \cos(2\pi k/N)} e^{2\pi ijk/N} = \sum_{\alpha \equiv \pm j \pmod{N}} (-i)^\alpha J_\alpha(t). \tag{30}$$

It is an open question if there exists a time $t \in \mathbb{R}^+$ such that for all $j \in \mathbb{Z}_N$ we have $|\psi_j(t)|^2 = 1/N$, i.e., uniformity is achieved at time t . For $N = 2, 3, 4$, it is known that *instantaneous exact uniform mixing* is achieved (see Refs. 6 and 16).

4. THE FINITE PATH

4.1. Classical Process

Let A be the *normalized* adjacency matrix of the finite path, where A is a stochastic matrix where the probability transitions are proportional to the degrees of the vertices. Let $H = A - I$ be its Laplacian. Then, the Kolmogorov equation, in this case, is

$$P'_j(t) = \frac{1}{2}P_{j-1}(t) - P_j(t) + \frac{1}{2}P_{j+1}(t), \quad (31)$$

for $0 < j < N$, with initial condition $P_j(0) = \delta_{j,0}$ and boundary conditions

$$P'_0(t) = P_1(t) - P_0(t), \quad P'_N(t) = P_{N-1}(t) - P_N(t). \quad (32)$$

The Laplace transform of Eq. (31) is

$$\hat{P}_{j+1}(s) - 2(s+1)\hat{P}_j(s) + \hat{P}_{j-1}(s) = 0, \quad 0 < j < N, \quad (33)$$

and two boundary equations $(1+s)\hat{P}_0(s) - 1 = \hat{P}_1(s)$, and $(1+s)\hat{P}_N(s) = \hat{P}_{N-1}(s)$. A guess of the solution is

$$\hat{P}_j(s) = Aq_+^j + Bq_-^j, \quad 0 \leq j \leq N, \quad (34)$$

where $q_{\pm} = (s+1) \pm \sqrt{(s+1)^2 - 1}$. The boundary equations give $B - A = 2/(q_+ - q_-)$ and $A = Bq_-^{2N}$. Combining these last two equations, we get

$$A = \frac{2}{(q_+ - q_-)} \frac{q_-^{2N}}{(1 - q_-^{2N})}. \quad (35)$$

Thus, for $j = 0, 1, \dots, N$,

$$\begin{aligned} \hat{P}_j(s) &= Aq_+^j + Bq_-^j = A(q_+^j + q_+^{2N-j}) = \frac{2}{(q_+ - q_-)} \frac{(q_-^j + q_-^{2N-j})}{(1 - q_-^{2N})} \\ &= \frac{2}{(q_+ - q_-)} \sum_{k=0}^{\infty} (q_-^{2Nk+j} + q_-^{2N(k+1)-j}). \end{aligned}$$

The inverse Laplace transform Eq. (4), after shifting, yields, for $j = 0, 1, \dots, N$,

$$P_j(t) = \sum_{k=0}^{\infty} e^{-t} [I_{2Nk+j}(t) + I_{2N(k+1)-j}(t)] = \sum_{\alpha \equiv \pm j \pmod{2N}} e^{-t} I_{\alpha}(t). \quad (36)$$

4.2. Quantum Process

The Schrödinger equation for the finite path is

$$i\psi'_j(t) = \frac{1}{2}\psi_{j-1}(t) + \frac{1}{2}\psi_{j+1}(t), \quad (37)$$

for $0 < j < N$, with initial condition $\psi_j(0) = \delta_{0,j}$ and boundary conditions

$$i\psi'_0(t) = \psi_1(t), \quad i\psi'_N(t) = \psi_{N-1}(t). \quad (38)$$

The Laplace transform of Eq. (37) is

$$\hat{\psi}_{j+1}(s) - 2is\hat{\psi}_j(s) + \hat{\psi}_{j-1}(s) = 0, \quad 0 < j < N, \quad (39)$$

and two boundary equations $is\hat{\psi}_0(s) - i = \hat{\psi}_1(s)$, and $is\hat{\psi}_N(s) = \hat{\psi}_{N-1}(s)$. The solutions of $q^2 - 2isq + 1$ are $q_{\pm} = i(s \pm \sqrt{s^2 + 1})$. A guess of the solution is

$$\hat{\psi}_j(s) = Aq_+^j + Bq_-^j, \quad 0 \leq j \leq N. \quad (40)$$

From the boundary equations, we get $B - A = 2i/(q_+ - q_-)$ and $B = Aq_+^{2N}$. Thus,

$$A = \frac{2i}{(q_+ - q_-)} \frac{q_-^{2N}}{(1 - q_-^{2N})}. \quad (41)$$

For $j = 0, 1, \dots, N$,

$$\begin{aligned} \hat{\psi}_j(s) &= Aq_+^j + Bq_-^j = A(q_+^j + q_+^{2N-j}) \\ &= \frac{2i}{(q_+ - q_-)} \frac{(q_-^j + q_-^{2N-j})}{(1 - q_-^{2N})} = \frac{2i}{(q_+ - q_-)} \sum_{k=0}^{\infty} (q_-^{2Nk+j} + q_-^{2(k+1)N-j}) \\ &= \sum_{k=0}^{\infty} \left[\frac{((-i)(\sqrt{s^2 + 1} - s))^{2Nk+j}}{\sqrt{s^2 + 1}} + \frac{((-i)(\sqrt{s^2 + 1} - s))^{2(k+1)N-j}}{\sqrt{s^2 + 1}} \right]. \end{aligned}$$

The inverse Laplace transform Eq. (5) yields, for $j=0, 1, \dots, N$,

$$\begin{aligned}\psi_j(t) &= \sum_{k=0}^{\infty} \left[(-i)^{2Nk+j} J_{2Nk+j}(t) + (-i)^{2N(k+1)-j} J_{2N(k+1)-j}(t) \right] \\ &= \sum_{\alpha \equiv \pm j \pmod{2N}} (-i)^\alpha J_\alpha(t).\end{aligned}\quad (42)$$

4.3. Spectral Analysis

The spectrum of a path on n vertices is given by Spitzer.⁽¹⁷⁾ For $j \in \{0, 1, \dots, N\}$, the eigenvalue λ_j and its eigenvector v_j are given by

$$\lambda_j = \cos\left(\frac{(j+1)\pi}{N+2}\right), \quad v_j(\ell) = \sqrt{\frac{2}{N+2}} \sin\left(\frac{(j+1)\pi}{N+2}(\ell+1)\right).\quad (43)$$

The probability of measuring vertex 0 at time t is given by

$$P_0(t) = \frac{4}{(N+2)^2} \sum_{j,k} \sin^2\left(\frac{(j+1)\pi}{N+2}\right) \sin^2\left(\frac{(k+1)\pi}{N+2}\right) e^{-it(\lambda_j - \lambda_k)}.\quad (44)$$

Since all eigenvalues are distinct, the *average probability* of measuring the starting vertex 0 is

$$\begin{aligned}\bar{P}(0) &= \frac{4}{(N+2)^2} \sum_{j,k} \sin^2\left(\frac{(j+1)\pi}{N+2}\right) \sin^2\left(\frac{(k+1)\pi}{N+2}\right) \\ &\quad \times \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{-it(\lambda_j - \lambda_k)} dt \\ &= \frac{4}{(N+2)^2} \sum_j \sin^4\left(\frac{(j+1)\pi}{N+2}\right).\end{aligned}$$

Equating this with Eq. (42), we obtain a Bessel-like equation:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left| \sum_{a \equiv 0 \pmod{2N}} (-i)^a J_a(t) \right|^2 dt = \frac{4}{(N+2)^2} \sum_{k=0}^N \sin^4\left(\frac{(k+1)\pi}{N+2}\right)\quad (45)$$

5. CONCLUSIONS

This expository survey reviews equations for the continuous-time quantum walks on one-dimensional lattices. The focus was on analysis based on the Laplace transform which works directly with the stochastic recurrences. It would be interesting to extend this analysis to higher-dimensional or to regular graph-theoretic settings. Another interesting direction is to consider lattices with defects and weighted graphs.⁽⁹⁾

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