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# STABILITY ANALYSIS OF PHYSICS-INFORMED NEURAL NETWORKS FOR STIFF LINEAR DIFFERENTIAL EQUATIONS

#### A PREPRINT

### Gianluca Fabiani $^1$ , Erik Bollt $^2$ , Constantinos Siettos $^3,^\ast$ , Athanasios N. Yannacopoulos $^{4,\ast}$

(1) Modelling Engineering Risk and Complexity, *Scuola Superiore Meridionale*, Naples 80138, Italy

 $(2)$ Dept. of Electrical and Computer Engineering, Clarkson University, Potsdam, NY, USA

(3)Dipartimento di Matematica e Applicazioni "Renato Caccioppoli", *Universita degli Studi di Napoli ` Federico II*, Naples 80126, Italy

(4)Department of Statistics, *Athens University of Economics and Business*, Greece

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#### ABSTRACT

We present a stability analysis of Physics-Informed Neural Networks (PINNs) coupled with random projections, for the numerical solution of (stiff) linear differential equations. For our analysis, we consider systems of linear ODEs, and linear parabolic PDEs. We prove that properly designed PINNs offer consistent and asymptotically stable numerical schemes, thus convergent schemes. In particular, we prove that multi-collocation random projection PINNs guarantee asymptotic stability for very high stiffness and that single-collocation PINNs are A-stable. To assess the performance of the PINNs in terms of both numerical approximation accuracy and computational cost, we compare it with other implicit schemes and in particular backward Euler, the midpoint, trapezoidal (Crank-Nikolson), the 2-stage Gauss scheme and the 2 and 3 stages Radau schemes. We show that the proposed PINNs outperform the above traditional schemes, in both numerical approximation accuracy and importantly computational cost, for a wide range of step sizes.

#### *K*eywords

Physics-Informed Neural Networks, Random Projections, Stability Analysis, stiff linear ODEs, Linear Parabolic PDEs

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## 1 Introduction

The use of machine learning (ML) for the solution of the forward problem, i.e. the numerical solution of (time-dependent) differential equations can be traced back to the '90s [\[35,](#page-17-0) [38,](#page-17-1) [19,](#page-17-2) [34,](#page-17-3) [21\]](#page-17-4). More recently, theoretical and technological advances have bloomed research activity in the field. Physics-Informed (Deep) Neural Networks (PINNs) [\[45,](#page-18-0) [32,](#page-17-5) [36,](#page-17-6) [29,](#page-17-7) [63,](#page-18-1) [62,](#page-18-2) [43,](#page-18-3) [57,](#page-18-4) [56\]](#page-18-5) represent the largest piece of the pie. Other ML schemes, include Gaussian Processes (GPs) [\[46,](#page-18-6) [47,](#page-18-7) [41,](#page-17-8) [5,](#page-16-0) [6\]](#page-16-1), Long Short Term Memory (LSTM) networks [\[4,](#page-16-2) [37\]](#page-17-9), generative adversarial networks (GANs)[\[59\]](#page-18-8), and various versions of random projection neural networks (RPNNs) [\[60,](#page-18-9) [11,](#page-16-3) [13,](#page-16-4) [2,](#page-16-5) [8,](#page-16-6) [9,](#page-16-7) [52,](#page-18-10) [7,](#page-16-8) [15,](#page-16-9) [58\]](#page-18-11), neural operators [\[3\]](#page-16-10). and the random feature model for learning the solution operator of PDEs [\[39,](#page-17-10) [40\]](#page-17-11). Recently, in [\[15\]](#page-16-9), it has been shown for the first time that, for several problems of stiff systems of ODEs and DAEs, bridging RPNNs with state-of-the-art numerical analysis and continuation techniques, can outperform state-of-the-art stiff time-adaptive integrators. In [\[33\]](#page-17-12), it has been given a stability and convergence analysis of PINNs coupled with linear multistep integrators for the solution of the *inverse* problem (i.e., learning of the

<sup>∗</sup>Corresponding authors, emails: constantinos.siettos@unina.it, ayannaco@aueb.gr

laws of dynamics from data) for stochastic systems. However, to the best of our knowledge, there is no study providing a systematic stability analysis of PINNs for the solution of the *forward* problem. Due to the inherent nonlinearities of these schemes, and the optimization procedure used for their training, stability is usually demonstrated only via numerical experiments [\[15\]](#page-16-9).

Here, we present a linear stability analysis of PINNs with random activation functions. We prove that properly designed PINNs with random Gaussian kernels, are asymptotically stable for the solution of linear stiff differential equations with the region of stability being precisely the left-half plane. We furthermore demonstrate that the PINNs are consistent schemes, thus convergent. We prove asymptotic stability for both multi-collocation and single collocation schemes. Our theoretical results are illustrated via numerical solutions of ODEs and discretized linear parabolic PDEs. Furthermore, we compare the numerical approximation accuracy and computational cost of the proposed PINNs with traditional implicit Runge-Kutta schemes for various step sizes. Our results demonstrate that the proposed PINN scheme outperforms the other implicit schemes in many scenarios.

The structure of the paper is as follows: In Section [2,](#page-1-0) we give in a nutshell the concept of PINNs for the solution of the forward problem for ODEs and the concept of random projection neural networks. In Section [3,](#page-2-0) we prove stability and consistency properties and in section 4, we provide some numerical examples.

# <span id="page-1-0"></span>2 PINNs for the solution of ODEs and Random Projection Neural Networks

Briefly, the physics-informed neural network (PINN) approach [\[45,](#page-18-0) [32\]](#page-17-5) for the solution of initial value problem (IVP), with  $d$  variables, of the form:

$$
\frac{d\boldsymbol{u}(t)}{dt} = \boldsymbol{f}(t, \boldsymbol{u}(t)), \qquad t \in [t_0, T], \qquad \boldsymbol{u}(t_0) = \boldsymbol{u}_0,\tag{1}
$$

approximates  $\bm u(t):\mathbb R\to\mathbb R^d$  with a trial solution  $\hat{\bm u}(t)\in C^1([t_0,T];\mathbb R^d)$  expressed as a feedforward neural network (FNN).  $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$  is a vector field. Considering a single layer FNN with a linear output activation element,  $\hat{u}(t)$  can be written, in matrix form, as:

$$
\hat{\mathbf{u}}(t, V, W, \mathbf{b}, \boldsymbol{\beta}) = W^T \Phi \left( V^T t + \mathbf{b}; P \right) + \boldsymbol{\beta}, \tag{2}
$$

where  $\Phi(t;P): \mathbb{R} \times \mathbb{R}^{N_p} \to \mathbb{R}^N$  denotes a vector function with elements appropriately parametrized basis functions  $\phi_j(t; p_j)$ ,  $V \in \mathbb{R}^{1 \times N}$  is the matrix with elements the weights  $v_j \in \mathbb{R}$  between the time input and the hidden layer,  $W \in \mathbb{R}^{N \times d}$  is the matrix with column vectors the weights  $w_{j,k}$  between the hidden and the k-th node in the output layer,  $b \in \mathbb{R}^N$  is the column vector with biases  $b_j$  of the hidden layer,  $\beta \in \mathbb{R}^d$  is the vector of biases  $\beta_k$  of the k-th output node and P denotes the hyperparameters of the actication functions and training algorithm. Then the solution of the forward problem reduces to the solution of the following optimization problem:

<span id="page-1-1"></span>
$$
\underset{V,W,\boldsymbol{b},\boldsymbol{\beta},P}{\arg\min} \left( \sum_{i=1}^{M} \left\| \frac{d\hat{\boldsymbol{u}}(t_i)}{dt} - \boldsymbol{f}(t_i, \hat{\boldsymbol{u}}(t_i)) \right\|^2 + \|\hat{\boldsymbol{u}}(t_0) - \boldsymbol{u}_0\| \right), \tag{3}
$$

on M collocation points  $c_i: t_0 < c_1 < c_2 < \cdots < c_i < \cdots < c_M < T$  in  $[t_0, T]$ , where the solution is sought.

Here, for the solution of the above system of ODEs, we consider PINNs with random projections (PIRPNNs) as proposed in [\[15\]](#page-16-9), where the activation functions are Gaussian radial basis functions with randomly selected centers and shape parameters. In this proposed scheme, the weights between the input and hidden layer V are set to ones, and the biases of the hidden and output layer are set to zeros.

Random projection neural networks (RPNNs) have been used for the solution of both the inverse [\[49,](#page-18-12) [11,](#page-16-3) [17,](#page-16-11) [14,](#page-16-12) [10,](#page-16-13) [16\]](#page-16-14) and forward problems  $[60, 55, 61, 11, 13, 2, 8, 9, 52, 7, 15, 58]$  in differential equations. The idea behind them, i.e., the use of randomly parametrized activation functions for single layer structures to simplify computations for the approximation of functions, can be traced back to the early '60s in the celebrated work of Rosenblatt [\[48\]](#page-18-15). It has been demonstrated, that such RPNNs are universal approximations for continuous functions (see e.g. [\[53,](#page-18-16) [1,](#page-16-15) [27,](#page-17-13) [44,](#page-18-17) [22\]](#page-17-14)). The various types of RPNNs that have been proposed during the years, such as Random Vector Functional Link Networks (RVFLNs) [\[27,](#page-17-13) [26\]](#page-17-15), Echo-State Neural Networks/Reservoir Computing [\[28\]](#page-17-16), and the so-called Extreme Learning Machines [\[25,](#page-17-17) [24\]](#page-17-18), share common concepts. A fundamental work, linking the above conceptually equivalent approaches is the celebrated Johnson and Lindenstrass (JL) Lemma (1984) [\[30\]](#page-17-19), which is but one of the various choices for approximating nonlinear functions. As it has been shown (see e.g.,  $[1, 42, 44, 22, 20]$  $[1, 42, 44, 22, 20]$  $[1, 42, 44, 22, 20]$  $[1, 42, 44, 22, 20]$  $[1, 42, 44, 22, 20]$  $[1, 42, 44, 22, 20]$  $[1, 42, 44, 22, 20]$  $[1, 42, 44, 22, 20]$  $[1, 42, 44, 22, 20]$ ), appropriately constructed nonlinear random projections may outperform such simple linear random projections. More recently, Rahimi and Recht [\[44\]](#page-18-17) demonstrated a universal approximation theorem for RPNNs. For a review on random projection neural networks, see [\[50,](#page-18-19) [15\]](#page-16-9).

The trial solution for the  $u^{(k)}$ -th component of the solution, with the proposed PINN with random projections can be written in the form:

<span id="page-2-2"></span>
$$
\hat{u}^{(k)}(t, \mathbf{w}_k) = u_0^{(k)} + (t - t_0) \sum_{j=1}^N w_{j,k} \phi_j(t; \alpha_j, \tau_j), \qquad \phi_j(t; \alpha_j, \tau_j) = \exp(-\alpha_j (t - \tau_j)^2), \tag{4}
$$

where  $\hat{u} = (\hat{u}^{(1)}, \dots, \hat{u}^{(d)}) \in \mathbb{R}^d$ , the basis function  $\phi_j$  are Gaussian radial basis functions (RBFs) with shape parameters  $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$  and centers  $\tau = (\tau_1, \tau_2, \ldots, \tau_N)$ ;  $w_{j,k}$  are the N unknown weights to be optimized. Note that the above trial function satisfies explicitly the initial condition at  $t = t_0$ . In the proposed scheme the hyperparameters are i.i.d. distributed. The shape parameters are drawn randomly from a uniform distribution across the interval, say,  $\mathcal{U} = \begin{bmatrix} 0 & \alpha_u \end{bmatrix}$  and the centers  $\tau_i$  of the RBFs are drawn randomly from a uniform distribution in the interval  $[t_0, T]$ .

For M collocation points, the minimization of the loss function in [\(3\)](#page-1-1) is performed over the  $Md$  nonlinear residuals  $F_q$ :

$$
F_q(W) = \frac{d\hat{u}^{(\ell)}}{dc_i}(c_i, \boldsymbol{w}_{\ell}) - f_{\ell}(c_i, \hat{u}^{(1)}(c_i, \boldsymbol{w}_1), \dots, \hat{u}^{(d)}(c_i, \boldsymbol{w}_d)),
$$
\n(5)

where  $q = i + (\ell - 1)d$ ,  $\ell = 1, 2, \ldots d$ ,  $i = 1, 2, \ldots M$ . The solution of the above non-linear least square problem can be obtained, e.g. with Newton-type iterations such as quasi-Newton, Gauss-Newton. For example, by setting  $\mathbf{F} = [F_1(W), \ldots, F_1(W), \ldots, F_{Md}(W)]$ , the update  $dW(\nu)$ , at the v-th iteration of the Newton scheme, is computed by the solution of the linearized system [\[15\]](#page-16-9):

<span id="page-2-1"></span>
$$
\nabla_{W^{(\nu)}} \boldsymbol{F} dW^{(\nu)} = -\boldsymbol{F}(W^{(\nu)}),\tag{6}
$$

where  $\nabla_{W^{(\nu)}}\bm F\in\mathbb R^{Md\times dN}$  is the Jacobian matrix of  $\bm F$  with respect to  $W^{(\nu)}$ . The solution of Eq. [\(6\)](#page-2-1) can be efficiently computed using, for example, truncated SVD decomposition or QR factorization with regularization  $[15, 12]$  $[15, 12]$ .

In general, the collocations points  $c_i$  are different from the centers  $\tau_j$  of the RBFs, with  $M < N$ , i.e., there are more neurons/nodes than collocation points. This choice (also called collocation technique of Kansa [\[31\]](#page-17-21)), leads to unsymmetrical interpolation matrices, which as it has been shown numerically, result to a better numerical performance with respect to the symmetrical collocation in which the collocation points coincide with the centers of the RBFs (see for example [\[23,](#page-17-22) [51\]](#page-18-20)). Moreover, the convergence rate of such PINNs for systems of ODEs has been shown in [\[15\]](#page-16-9), through their connection with the Picard-Lindelöf's scheme.

At the following section, we will convey the stability analysis for both multi-collocation and single collocation  $(M = 1)$  schemes with RBFs in Eq. [\(4\)](#page-2-2). We will assume that (i) the hyperparameters  $\alpha^{(\ell)}$  =  $[\alpha_1^{(\ell)}, \alpha_2^{(\ell)}, \dots, \alpha_j^{(\ell)}, \dots, \alpha_N^{(\ell)}]^T$ , across each sub-interval  $[t_{\ell-1} \quad t_{\ell}]$  are drawn randomly from the same uniform distribution, say,  $\mathcal{U} = \begin{bmatrix} 0 & \alpha_u \end{bmatrix}$ ; (ii) the centers  $\tau_i$  of the RBFs are drawn randomly from a uniform distribution in  $[t_{\ell-1}, t_{\ell}]$  and (iii) the M collocations points are equally spaced in  $[t_{\ell-1}, t_{\ell}]$ .

#### <span id="page-2-0"></span>3 Linear Stability Analysis of Physics-Informed Neural Networks

We will consider linear ODEs of the form

<span id="page-2-3"></span>
$$
\frac{d\boldsymbol{u}(t)}{dt} = A\boldsymbol{u}, \qquad \boldsymbol{u}(t_0) = \boldsymbol{u}_0,\tag{7}
$$

where  $u \in C^1(0,T;\mathbb{R}^d)$ ,  $A \in \mathbb{R}^{d \times d}$  is a diagonalizable matrix, and  $u_0 \in \mathbb{R}^d$  is a given initial condition.

Note that the proposed PINN numerical scheme, is probabilistic in nature. Hence, it produces, at each step  $\ell$ , approximations of the solution which are random variables. The numerical method produces a sequence  $\{\hat{u}_\ell\}\subset\mathbb{R}^d$ of random vectors, where, for each  $\ell \in \mathbb{N}$ ,  $\hat{u}_{\ell}$  is an approximation of the solution of the system [\(4\)](#page-2-2) in the interval  $[t_{\ell}, t_{\ell+1})$ . The proposed PINN numerical scheme may then be expressed in compact form

<span id="page-2-4"></span>
$$
\boldsymbol{u}_{\ell+1}(\omega) = S(\Pi, \omega) \boldsymbol{u}_{\ell}(\omega), \tag{8}
$$

where we use  $\omega$  to emphasize that the output of the proposed numerical method is random (in our case on account of the random samples of  $\alpha$  and  $\tau$ ) and  $S(\Pi,\omega)$  is a random matrix depending on a set of parameters  $\Pi$ , which characterize the method (in our case  $a_U$ , h etc).

As an intermediate, important, step in our analysis, we first consider the stability analysis linear scalar ODEs. Then, we will use these results to extend our analysis to the multidimensional system [\(7\)](#page-2-3).

# 3.1 The case of linear scalar ODEs

In this section we consider the problem

<span id="page-3-1"></span>
$$
\frac{du(t)}{dt} = \lambda u, \qquad u(t_0) = u_0,\tag{9}
$$

In this case [\(8\)](#page-2-4) reduces to a scalar equation where  $S := S(\Pi, \omega) \in \mathbb{R}$ . The system is stable if  $|S| < 1$  and unstable if  $|S| > 1.$ 

Next, we will analyze both the multi-collocation and single collocation schemes. We will first prove that the multicollocation PINN scheme is stable for very large stiffness, and A-stable for the single collocation case. Then, we prove the consistency of the scheme. For analysis, we choose fixed equidistant collocation points  $c_i \in (t_{\ell-1}, t_\ell],$  $i = 1, \cdots, M$ , defined as:

<span id="page-3-0"></span>
$$
c_i = t_{\ell-1} + (t_{\ell} - t_{\ell-1})\zeta_i = t_{\ell-1} + h\zeta_i, \quad \zeta_i = \frac{i}{M}.
$$
\n(10)

We also define for each  $\ell$ , the matrix

$$
\Psi^{(\ell)} = \left(\psi_{ji}^{(\ell)}\right)_{j=1,\cdots,N}^{i=1,\cdots,M} \in \mathbb{R}^{N \times M},\tag{11}
$$

by

$$
\psi_{ji}^{(\ell)} := \psi_j(c_i, \alpha_j^{(\ell)}) := \phi_j(t, \alpha_j^{(\ell)}) + (c_i - t_{\ell-1})\phi_j'(c_i, \alpha_j^{(\ell)}) - \lambda(c_i - t_{\ell-1})\phi_j(c_i, \alpha_j^{(\ell)}),
$$
(12)

as well as the vector

$$
\Phi^{(\ell)}(t) = (\phi(t, \alpha_1^{(\ell)}, \tau_1^{(\ell)}), \cdots, \phi(t, \alpha_N^{(\ell)}, \tau_N^{(\ell)}))^T = (\phi_1^{(\ell)}(t), \cdots, \phi_N^{(\ell)}(t))^T \in \mathbb{R}^{N \times 1}.
$$
\n(13)

We will also use the notation  $\hat{u}_{\ell} := \hat{u}(t_{\ell})$  (which does not necessarily coincide with  $u(t_{\ell})$ , unless  $\ell = 0$ , i.e., at the initial condition.

The training of the PINN will be done by choosing the weights as minimizers of the (family) of loss functionals

$$
\mathcal{L}_{\delta}^{(\ell)}(w) = \frac{1}{2} \sum_{i=1}^{M} (\hat{u}'(c_i; w) - \lambda \hat{u}(c_i; w))^2 + \frac{\delta}{2} \sum_{j=1}^{N} |w_j|^2,
$$

for  $c_i \in (t_{\ell-1}, t_\ell]$  as in [\(10\)](#page-3-0) above. The second term in  $\mathcal{L}_{\delta}^{(\ell)}$  $\delta^{\left(\epsilon\right)}$  is a regularization term, which allows for a unique solution to the minimization problem.

<span id="page-3-2"></span>**Proposition 3.1.** The regularized multi-collocation scheme, provides for each  $t \in (t_{\ell-1}, t_{\ell})$  the solution for each of the collocation points  $c_i \in (t_{\ell-1}, t_{\ell})$  (defined as in [\(10\)](#page-3-0)):

<span id="page-3-3"></span>
$$
\hat{u}_{\ell}(c_i) = \hat{u}_{\ell-1} \left( 1 + \lambda h \zeta_i (\Phi_i^{(\ell)})^T (\Psi^{(\ell)} (\Psi^{(\ell)})^T + \delta I)^{-1} \Psi^{(\ell)} \mathbf{1}_{M \times 1} \right), i = 1, \cdots, M, \ \delta > 0.
$$
 (14)

In the limit as  $\delta \to 0^+$  this reduces to

$$
u(t) = \hat{u}_{\ell-1} + (t - t_{\ell-1})(\Phi^{(\ell)})^T(t)w,
$$

where  $w$  is the least-squares solution of the collocation system

$$
\sum_{j=1}^{N} w_j^{(\ell)} \psi_{ji}^{(\ell)} = \lambda u(t_{\ell-1}), \ \ i = 1, \cdots, M \iff (\Psi_{N \times M}^{(\ell)})^T \mathbf{w}_{N \times 1}^{(\ell)} = \lambda u(t_{\ell-1}) \mathbb{1}_{1 \times M}. \tag{15}
$$

*Proof.* For each interval,  $[t_{\ell-1}, t_{\ell}]$  consider the expansion [\(4\)](#page-2-2). We calculate the time derivative  $\hat{u}'(t)$  in this interval to obtain:  $\mathbf{v}$ 

$$
\hat{u}'(t) = \sum_{j=1}^{N} w_j^{(\ell)} \hat{\psi}_j(t; \alpha_j^{(\ell)}), \hat{\psi}_j(t, \alpha_j^{(\ell)}) = \phi_j(t, \alpha_j^{(\ell)}) + (t - t_{\ell-1})\phi_j'(t, \alpha_j^{(\ell)}).
$$
\n(16)

Calculate the differential equation on each of the collocation points  $c_i \in (t_{\ell-1}, t_{\ell}]$ :

$$
\hat{u}'(c_i) - \lambda \hat{u}(c_i) = \sum_{j=1}^{N} w_j^{(\ell)} \left( \hat{\psi}_j(c_i, \alpha_j^{(\ell)}) - \lambda(c_i - t_{\ell-1}) \phi_j(c_i, \alpha_j^{(\ell)}) \right) - \lambda \hat{u}_{\ell-1},\tag{17}
$$

which upon defining

$$
\psi_j(t, \alpha_j^{(\ell)})] := \hat{\psi}_j(t, \alpha_j^{(\ell)}) - \lambda(t - t_{\ell-1})\phi_j(t, \alpha_j^{(\ell)}) = \phi_j(t, \alpha_j^{(\ell)}) + (t - t_{\ell-1})\phi_j'(t, \alpha_j^{(\ell)}) - \lambda(t - t_{\ell-1})\phi_j(t, \alpha_j^{(\ell)}),
$$
\n(18)

reduces to

$$
\hat{u}'(c_i) - \lambda \hat{u}(c_i) = \sum_{j=1}^{N} w_j^{(\ell)} \psi_j(c_i, \alpha_j^{(\ell)}) - \lambda \hat{u}_{\ell-1}, \ \ i = 1, \cdots, M. \tag{19}
$$

Defining  $\psi_{ji}^{(\ell)} := \psi_j(c_i, \alpha_j^{(\ell)})$  $j^{(\ell)}$ ), the above is abbreviated as

<span id="page-4-0"></span>
$$
E_i := \hat{u}'(c_i) - \lambda \hat{u}(c_i) = \sum_{j=1}^N w_j^{(\ell)} \psi_{ji}^{(\ell)} - \lambda \hat{u}_{\ell-1} = [(\Psi^{(\ell)})^T w_{N \times 1}]_i - \lambda \hat{u}_{\ell-1}.
$$
 (20)

Note that [\(20\)](#page-4-0) can be expressed conveniently in matrix form in terms of  $E = (E_1, \dots, E_M)^T$  as

$$
E = (\Psi^{(\ell)})^T w_{N \times 1} - \lambda \hat{u}_{\ell-1} \mathbf{1}_{M \times 1}.
$$
\n(21)

We now calculate the loss function [\(3\)](#page-1-1), over all  $E_i$ ,  $i = 1, 2, \ldots, M$ .

$$
\mathcal{L}_0 = \frac{1}{2} \sum_{i=1}^M \left( \hat{u}'(c_i) - \lambda \hat{u}(c_i) \right)^2 = \frac{1}{2} \left\| (\Psi^{(\ell)})^T w_{N \times 1} - \lambda \hat{u}_{\ell-1} \mathbf{1}_{M \times 1} \right\|_{\mathbb{R}^M}^2.
$$
 (22)

To simplify the notation, we set  $A_\ell := (\Psi^{(\ell)})^T$ . In this notation, for each  $\ell$ , the solution is given in terms of the set of weights  $w_{N}^{(\ell)}$  $\frac{K}{N\times1}$ , which is a solution of the linear least squares problem

$$
w_{N\times 1}^{(\ell)} \in \arg\min \frac{1}{2} \|A_{\ell}w_{N\times 1} - \lambda \hat{u}_{\ell-1}\mathbf{1}_{M\times 1}\|^2.
$$
 (23)

For the case that  $\delta > 0$ , we proceed with the first order conditions of the function  $\mathcal{L}_{\delta} = \mathcal{L}_0 + \frac{\delta}{2} ||w||^2$ . Let us differentiate the above with respect to  $w_q^{(\ell)}$ ,  $q = 1, \cdots, N$ . This gives

$$
\frac{\partial \mathcal{L}_{\delta}}{\partial w_{q}^{(\ell)}} = \sum_{i=1}^{M} \left( \sum_{j=1}^{N} w_{j}^{(\ell)} \psi_{ji}^{(\ell)} - \lambda \hat{u}_{\ell-1} \right) \psi_{qi}^{(\ell)} + \delta w_{q}^{(\ell)} = 0, \ q = 1, \cdots, N. \tag{24}
$$

This can be expressed in a compact matrix form, in terms of the matrix  $\Psi^{(\ell)} = \Psi_{N \times M}^{(\ell)} = (\psi_{ji}^{(\ell)})_{j=1,\dots,N}^{i=1,\dots,M} \in \mathbb{R}^{N \times M}$ as

$$
\left(\Psi^{(\ell)}(\Psi^{(\ell)})^T + \delta I\right)w^{(\ell)} = \Psi^{(\ell)}\mathbf{1}_{M\times 1}\lambda\hat{u}_{\ell-1},\tag{25}
$$

which yields a solution for the regularized problem

<span id="page-4-1"></span>
$$
w_{N\times 1}^{(\ell)} = ((\Psi^{(\ell)})^T)^{\dagger}_{\delta} \mathbf{1}_{M\times 1} \lambda \hat{u}_{\ell-1}, \quad ((\Psi^{(\ell)})^T)^{\dagger}_{\delta} := [\Psi^{(\ell)}(\Psi^{(\ell)})^T + \delta I]^{-1} \Psi^{(\ell)}.
$$
 (26)

Note that this result holds even if  $\Psi^{(\ell)}(\Psi^{(\ell)})^T$  is not invertible. Hence,

$$
u(t) = \hat{u}_{\ell-1} \bigg( 1 + \lambda (t - t_{\ell-1}) (\Phi^{(\ell)})^T(t) \bigg[ \Psi^{(\ell)} (\Psi^{(\ell)})^T + \delta I \bigg]^{-1} \Psi^{(\ell)} \mathbf{1}_{M \times 1} \bigg). \tag{27}
$$

The connection with the Moore-Penrose pseudo inverse comes by recalling the well known relation that for any matrix A:

$$
A^{\dagger} = \lim_{\delta \to 0^{+}} (A^{T} A + \delta I)^{-1} A^{T} = \lim_{\delta \to 0^{+}} A^{T} (A A^{T} + \delta I)^{-1}.
$$

The stated result arises by taking the transpose of  $(26)$ .

 $\Box$ 

#### 3.1.1 Stability analysis of PINNs for linear scalar ODEs

<span id="page-5-3"></span>We will first prove the following theorem.

**Theorem 3.1.** The PINN scheme given by [\(4\)](#page-2-2), with L sub-intervals of size  $h$ , N neurons, M collocation points in each sub-interval and  $\alpha_j^{(\ell)} = \frac{1}{2}$  $|\lambda|h^{-1}$  $\frac{|\lambda|h^{-1}}{1+|\lambda|h+|\lambda|^2h^2}\theta_j^{(\ell)}, \theta_j^{(\ell)} \sim \mathcal{U}(0,1)$  is asymptotically stable for  $\lambda h \to -\infty$ ,  $\forall i = 1,\ldots,M$ ,  $\forall \ell = 1,\ldots,L$ . The PINN is also asymptotically stable for  $\lambda h \to 0^-$  and unstable for  $\lambda h \to 0^+$ , for the linear scalar ODE [\(9\)](#page-3-1).

*Proof.* By Proposition [3.1](#page-3-2)

<span id="page-5-1"></span>
$$
\hat{u}_{\ell}(c_i) = S_i^{(\ell)}(\lambda h, \zeta_i) \hat{u}_{\ell-1}, \quad S_i^{(\ell)}(\lambda h, \zeta_i) = \left(1 + \lambda h \zeta_i (\Phi_i^{(\ell)})^T \left[\Psi^{(\ell)}(\Psi^{(\ell)})^T + \delta I\right]^{-1} \Psi^{(\ell)} \mathbf{1}_{M \times 1}\right). \tag{28}
$$

We will use the notation  $z = \lambda h$ . By considering the Gaussian RBFs as defined in [\(4\)](#page-2-2) and, setting the shape parameters as:

<span id="page-5-0"></span>
$$
\alpha_j^{(\ell)} = \frac{1}{2} \theta_j^{(\ell)} \alpha_U(|\lambda|, h), \quad \text{where} \quad \alpha_U(|\lambda|, h) := \frac{|\lambda|h^{-1}}{1 + |\lambda|h + |\lambda|^2 h^2}, \quad \theta_j^{(\ell)} \sim \mathcal{U}(0, 1), \tag{29}
$$

and rewriting the centers of the RBFs as:

$$
\tau_j^{(\ell)} = t_{\ell-1} + \xi_j^{(\ell)} h, \qquad \xi_j^{(\ell)} \sim \mathcal{U}(0, 1), \tag{30}
$$

the elements of the matrix  $\Psi_{N \times M}$ , are given by:

$$
\psi_{ji}^{(\ell)} = (1 - \lambda h \zeta_i + h^2 a_U(|\lambda|, h) \theta_j^{(\ell)} (\zeta_i - \xi_j^{(\ell)}) \zeta_i) \phi_{ji}^{(\ell)} =
$$
  
= 
$$
(1 - \lambda h \zeta_i + h^2 a_U(|\lambda|, h) \theta_j^{(\ell)} (\zeta_i - \xi_j^{(\ell)}) \zeta_i) \exp(-a_U(|\lambda|, h) h^2 \theta_j^{(\ell)} (\zeta_i - \xi_j^{(\ell)})^2).
$$
 (31)

We will choose  $a_U(|\lambda|, h)$  so that  $h^2a_U(|\lambda|, h) = \alpha(z)$  where  $z := \lambda h$  and then choose  $\alpha(z)$  in such a way as to satisfy

$$
\alpha(z) \to 0 \quad \text{as} \quad z \to \pm \infty, \text{ and, } \alpha(z) \to 0 \quad \text{as} \quad z \to 0. \tag{32}
$$

An example of such a choice is as in [\(29\)](#page-5-0), i.e.,

$$
a_U(\lambda, h) = \frac{|\lambda|h^{-1}}{1 + |\lambda|h + |\lambda|^2 h^2}, \text{leading to, } \alpha(z) := h^2 \frac{|\lambda|h^{-1}}{1 + |\lambda|h + |\lambda|^2 h^2} = \frac{|z|}{1 + |z| + |z|^2}.
$$
 (33)

Dropping the superscript  $\ell$  for ease of notation, we express

<span id="page-5-2"></span>
$$
\psi_{ji}(z) = \left(1 - z\zeta_i + \alpha(|z|)(\zeta_i - \xi_j)\theta_j\zeta_i\right)\phi_{ji}(z),\tag{34}
$$

where we have explicitly stated the dependence on  $z := \lambda h$  for emphasis. Moreover, the stability index becomes

$$
S_i(z) = \left(1 + \zeta_i z \Phi_i(z)^T \left[\Psi(z)\Psi(z)^T + \delta I\right]^{-1} \Psi(z) \mathbf{1}_{M \times 1}\right). \tag{35}
$$

We now consider the limits  $z \to 0^{\pm}$  and  $z \to \pm \infty$ .

(a) In the limit as  $z \to 0^{\pm}$  we easily see that

<span id="page-5-4"></span>
$$
\Phi_i^T(z) \to \mathbf{1}_{1 \times N}, \quad \Psi(z) \to \mathbf{1}_{N \times M}, \tag{36}
$$

and invoking the continuity of  $\Phi_i^T(z) \big[ \Psi(z) \Psi^T(z) + \delta I \big]^{-1}$ , we see that [\(28\)](#page-5-1) implies that  $S_i(z) \to 1^{\pm}$ . Hence, the scheme has the correct stability around 0.

(b) We now consider the limit as  $z \to -\infty$ . In this limit [\(34\)](#page-5-2) yields

$$
\psi_{ji}(z) \simeq \zeta_i |z|. \tag{37}
$$

Therefore,

$$
\Psi(z)\mathbf{1}_{M\times 1} \simeq |z| \begin{pmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_M \\ \zeta_1 & \zeta_2 & \cdots & \zeta_M \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_1 & \zeta_2 & \cdots & \zeta_M \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = |z| ||\zeta||_1 \mathbf{1}_{N\times 1}
$$
\n(38)

We also note that in the limit as  $z \to -\infty$  we have that

$$
\Phi_i^T(z) \simeq \mathbf{1}_{1 \times N}.\tag{39}
$$

The matrix  $\Psi_{\delta}(z) := [\Psi(z)\Psi^{T}(z) + \delta I]^{-1}$  is positive definite for all z. In the limit as  $z \to -\infty$ , we have that

$$
\Psi(z)\Psi^{T}(z) + \delta I \simeq |z|^{2} \|\zeta\|_{2}^{2} \mathbf{1}_{N \times N} + \delta I,
$$
\n(40)

which is clearly positive definite, hence so is

$$
\Psi_{\delta,\infty}^{\dagger} := \left[\Psi(z)\Psi^{T}(z) + \delta I\right]^{-1} \simeq \left[|z|^{2}\|\zeta\|_{2}^{2}\mathbf{1}_{N\times N} + \delta I\right]^{-1}.\tag{41}
$$

Hence,

<span id="page-6-0"></span>
$$
S_i(z) \simeq \left(1 - \|\zeta\|_1 \|\zeta\|_2^{-2} \zeta_i \underbrace{\mathbf{1}_{1 \times N} \left[\mathbf{1}_{N \times N} + \delta |z|^{-2} \|\zeta\|_2^{-2} I\right]^{-1} \mathbf{1}_{N \times 1}}_{:=I' \geq 0}\right). \tag{42}
$$

In the limit as  $z \to -\infty$ , we see that  $S_i(z) < 1$ . We now try to consider its value. To this end, we restate [\(42\)](#page-6-0) in more convenient form. We first note that

$$
\mathbf{1}_{N\times N} = \frac{1}{M} \mathbf{1}_{N\times M} \mathbf{1}_{M\times N}, \quad \mathbf{1}_{N\times 1} = \frac{1}{M} \mathbf{1}_{N\times M} \mathbf{1}_{M\times 1}.
$$
 (43)

Using, the above we obtain

<span id="page-6-1"></span>
$$
S_i(z) \simeq 1 - \|\zeta\|_1 \|\zeta\|_2^{-2} \zeta_i \mathbf{1}_{1 \times N} \left[\mathbf{1}_{N \times M} \mathbf{1}_{M \times N} + \delta M |z|^{-2} \|\zeta\|_2^{-2} I\right]^{-1} \mathbf{1}_{N \times M} \mathbf{1}_{M \times 1}.
$$
 (44)

If we take the limit as  $|z| \to \infty$  in [\(42\)](#page-6-0), then

$$
\lim_{|z| \to \infty} \left[ \mathbf{1}_{N \times M} \mathbf{1}_{M \times N} + \delta M |z|^{-2} ||\zeta||_2^{-2} I \right]^{-1} \mathbf{1}_{N \times M} = \lim_{\epsilon \to 0} \left[ \mathbf{1}_{N \times M} \mathbf{1}_{M \times N} + \epsilon I \right]^{-1} \mathbf{1}_{N \times M} = \mathbf{1}_{M \times N}^{\dagger}
$$
\n
$$
= \left[ Tr(\mathbf{1}_{M \times N}^T \mathbf{1}_{M \times N}) \right]^{-1} \mathbf{1}_{M \times N}^T = (NM)^{-1} \mathbf{1}_{N \times M},
$$
\n(45)

where we used the properties (and the alternative definition) of the Moore -Penrose pseudo inverse.

Note also that if A is a rank 1 matrix, it holds that  $A^{\dagger} = c^{-1}A^T$ , where  $c = Tr(A^T A)$ . Hence, [\(42\)](#page-6-0) (or equivalently [\(44\)](#page-6-1)) yields

$$
S_i(z) \to 1 - \|\zeta\|_1 \|\zeta\|_2^{-2} \zeta_i \mathbf{1}_{1 \times N} \mathbf{1}_{M \times N}^{\dagger} \mathbf{1}_{M \times 1} = 1 - \|\zeta\|_1 \|\zeta\|_2^{-2} \zeta_i (NM)^{-1} \mathbf{1}_{1 \times N} \mathbf{1}_{N \times M} \mathbf{1}_{M \times 1} = 1 - \|\zeta\|_1 \|\zeta\|_2^{-2} (NM)^{-1} (NM) \zeta_i = 1 - \|\zeta\|_1 \|\zeta\|_2^{-2} \zeta_i.
$$
 (46)

It remains to calculate  $\|\zeta\|_1 \|\zeta\|_2^{-2}$ . Since  $\zeta_i = i/M$  we have that

$$
\|\zeta\|_1 = \sum_{i=1}^M \zeta_i = \frac{1}{M} \sum_{i=1}^M i = \frac{M+1}{2}, \quad \|\zeta\|_2^2 = \sum_{i=1}^M \zeta_i^2 = \frac{M(2M+1)(M+1)}{6M^2}.
$$
 (47)

Combining the above, we find that

$$
S_i(z) \to S_{i,\infty} := 1 - i \frac{3}{2M + 1}.
$$
\n(48)

It can be easily shown that  $0 < S_{i,\infty} < 1$  for all values of M and i. Since  $i = 1, \dots, M$  we see that

$$
S_{i,\infty} \in (-0.5,0) \cup (0,1),\tag{49}
$$

with the infimum  $-0.5$  corresponds to the collocation point  $i = M$ , as  $M \to \infty$ ) whereas the supremum 1 corresponds to the first collocation point  $i = 1$  as  $M \to \infty$ ). Hence for any i, we have that in the limit as  $z \to -\infty$ ,  $S_i(z) \in (-0.5, 1)$ , hence, the multi-collocation PINN is asymptotically stable.  $(-0.5, 1)$ , hence, the multi-collocation PINN is asymptotically stable.

Remark 1. As also demonstrated by numerical simulations depicted in Figure [1,](#page-7-0) the multi-collocation PINN scheme is A-stable. The behaviour of the stability index  $S$  for various choices of  $N$  and  $M$  is illustrated in Figure [1.](#page-7-0) We recall the following definitions concerning A-Stability:

Definition 3.1 (A-Stability). A numerical method that solves ordinary differential equations is said to be *A-stable* if its region of absolute stability includes the entire left half of the complex plane.

<span id="page-6-2"></span>We will now prove the following result for the RPNN scheme with one collocation point ( $M = 1$ ) at  $\zeta_i = t_i$ .

**Proposition 3.2.** The proposed scheme for one collocation point  $(M = 1)$  is A- stable a.s. for every  $z < 0$ .



<span id="page-7-0"></span>Figure 1: Multi-collocation PINN scheme for the solution of the scalar linear ODE. Absolute stability region of the multicollocation PINN scheme (colored zone), with numerical simulations. We report the absolute stability function |S| in the range [0, 1]. The white zone indicates  $|S| > 1$  (unstable region). A comparison with the stability domains of explicit Euler, and Runge-Kutta methods of orders 2, 3 and 4 (contour lines) are also given. We used a mesh of 141 × 161 points in the complex region  $\{z = (\mathcal{R}e(z), \mathcal{I}m(z)) \in [-10, 4] \times [-8, 8]\}$ : in (a)-(d)  $M = 4$ ; in (b)-(e)  $M = 10$ ; in (c)-(f)  $M = 50$ ; we set  $N = 3M$ . We report maximum values of the stability function in the first row  $((a)-(c))$ , and mean values in the second row  $((d)-(f))$  over 2000 Monte-Carlo runs.

*Proof.* Using the results and notation of Proposition [3.1,](#page-3-2) and setting  $s = |z|$ , we see that for  $M = 1$  the stability function becomes

$$
S(s) = 1 - s \frac{\mathcal{N}(s)}{\mathcal{D}(s) + \delta},
$$
  
\n
$$
\mathcal{N}(s) = \sum_{j=1}^{N} (1 + s + \theta_j (1 - \xi_j) a(s)) e^{-a(s)\theta_j (1 - \xi_j)^2},
$$
  
\n
$$
\mathcal{D}(s) = \sum_{j=1}^{N} (1 + s + \theta_j (1 - \xi_j) a(s))^2 e^{-a(s)\theta_j (1 - \xi_j)^2}.
$$
\n(50)

Since  $\theta_j, \xi_j \sim U([0, 1])$ , it holds that  $0 \le \theta_j (1 - \xi_j) \le 1$  a.s., hence

$$
N(1+s)e^{-a(s)} \le N(s) \le N(1+s+a(s)), N(1+s)^2e^{-a(s)} \le D(s) \le N(1+s+a(s))^2,
$$
 (51)

and consequently we obtain the following bounds for the stability function

$$
1 - \frac{s(1+s+a(s))}{\frac{\delta}{N} + (1+s)^2 e^{-a(s)}} \le S(s) \le 1 - \frac{s(1+s)e^{-a(s)}}{\frac{\delta}{N} + (1+s+a(s))^2} < 1, \ a.s.
$$
 (52)

To guarantee stability if suffices to show that the lower limit is strictly greater than  $-1$ , i.e.

$$
s(1+s+a(s)) < 2(1+s)^2 e^{-a(s)} + \frac{2\delta}{N}.\tag{53}
$$

<span id="page-8-5"></span>Clearly, if we show the above for  $\delta = 0$ , then it holds for every  $\delta > 0$ . Hence it suffices to prove the inequality

<span id="page-8-1"></span>
$$
s(1+s+a(s)) < 2(1+s)^2 e^{-a(s)}.\tag{54}
$$

To show that, we note that  $a(s)$  satisfies the bound  $0 \le a(s) \le \frac{1}{3}$  (this function attains its maximum value at  $s = 1$ ). Therefore,

<span id="page-8-0"></span>
$$
s(1+s+a(s)) \le s(1+s+\frac{1}{3}) = s(\frac{4}{3}+s), \quad 2(1+s)^2 e^{-1/3} \le 2(1+s)^2 e^{-a(s)}.
$$
 (55)

Then, as  $a(s) = s/(1 + s + s^2)$ , [\(55\)](#page-8-0) implies that [\(54\)](#page-8-1) will hold as long as,

$$
s(\frac{4}{3}+s) < 2(1+s)^2 e^{-1/3} \iff (1-2e^{-1/3})s^2 + 4(\frac{1}{3}-e^{-1/3})s - 2e^{-1/3} < 0,\tag{56}
$$

which is true for all  $s = |z| > 0$ . Hence, it holds that

$$
-1 < S(z) < 1, \ \forall \ z < 0, \ a.s., \tag{57}
$$

 $\Box$ 

from which, as the above holds  $\forall z < 0$ , we obtain the A- stability of the scheme.

#### 3.1.2 Consistency of PINNs for linear scalar ODEs

**Theorem 3.2.** Let  $u(\cdot)$  be the exact solution of the linear scalar ODE [\(9\)](#page-3-1), for any  $\lambda < 0$ , and  $\hat{u}(\cdot)$  be the approximate solution provided by the multi-collocation PINN scheme given by [\(4\)](#page-2-2). Then,

$$
\mathcal{E}_i = u_\ell(c_i) - \hat{u}_\ell(c_i) \to 0, \text{ as } h \to 0, \forall \delta > 0, M.
$$

Moreover, in the limit as  $\delta \to 0$  or  $M \to \infty$  it satisfies the property

$$
\frac{\mathcal{E}_i}{h} \to 0 \text{ as } h \to 0,
$$

i.e. it is consistent for the solution of [\(7\)](#page-2-3) in the limit as  $M \to \infty$ .

*Proof.* We will check consistency at any collocation point. In order to ease the notation, and since we are focusing on a single interval  $[t_{\ell-1}, t_{\ell}]$  we will omit the supercripts  $(\ell)$  from the representation [\(14\)](#page-3-3), which now reads

$$
\hat{u}_{\ell}(c_i) = \left(1 + \lambda h \zeta_i \Phi_i^T (\Psi \Psi^T + \delta I)^{-1} \Psi \mathbf{1}_{M \times 1}\right) u(t_{\ell-1}),\tag{58}
$$

where as mentioned above we assume that at  $t_{\ell-1}$  we have the true solution  $u(t_{\ell-1})$  and not the approximate solution  $\hat{u}_{\ell-1}$  (as in [\(14\)](#page-3-3)).

We now estimate the error of the method at each collocation point by using

<span id="page-8-3"></span><span id="page-8-2"></span>
$$
\mathcal{E}_i = u_\ell(c_i) - \hat{u}_\ell(c_i) = u_\ell(c_i) - \left(1 + \lambda h \zeta_i \Phi_i^T (\Psi \Psi^T + \delta I)^{-1} \Psi \mathbf{1}_{M \times 1}\right) u_{\ell-1}.
$$
\n(59)

We now use Taylor's theorem around to point  $t_{\ell-1}$  to obtain for the true solution that

$$
u_{\ell}(c_i) = u(t_{\ell-1}) + h\zeta_i u'_{\ell-1} + O(h^2) = u_{\ell-1} + \lambda h\zeta_i u_{\ell-1} + O(h^2),
$$
  
where we used the fact that u solves the ODE, and the regularity of the solution. (60)

Combining  $(59)$  and  $(60)$  we obtain that

<span id="page-8-4"></span>
$$
\mathcal{E}_i = \lambda h \zeta_i \left( \underbrace{1 - \Phi_i^T (\Psi \Psi^T + \delta I)^{-1} \Psi \mathbf{1}_{M \times 1}}_{:=A(h)} \right) + O(h^2)
$$
\n(61)

As long as the limit  $\lim_{h\to 0^+} A(h)$  is finite then we can easily see from [\(61\)](#page-8-4) that  $\lim_{h\to 0^+} \mathcal{E}_i = 0$  from which consistency follows.

To show that  $A(h) \to c$ , finite, in the limit as  $h \to 0^+$ , we work in the same spirit as in the proof of Theorem [3.1.](#page-5-3) To facilitate the proof and the exposition we set  $z = \lambda h$  and keeping  $\lambda$  finite we take the limit as  $h \to 0^+$ , i.e. consider the limit as  $z \to 0$  (we will allow  $\lambda$  being either positive or negative, as we want to show consistency in the general case). We then express  $A(h)$  as

$$
A(z) = 1 - \Phi_i^T(z) (\Psi(z)\Psi^T(z) + \delta I)^{-1} \Psi(z) \mathbf{1}_{M \times 1}
$$
 (62)

Note that, using the notation  $a(z) = ha_U(\lambda, h)$ , where  $a(z)$  is chosen so that  $a(z) \to 0$  as  $z \to 0$ , the observation [\(36\)](#page-5-4) and the continuity of  $z \to (\Psi(z)\Psi^{T}(z) + \delta I)^{-1}$  we conclude that

$$
A(z) \to A_0(\delta, M) := 1 - \mathbf{1}_{1 \times N} (\mathbf{1}_{N \times M} \mathbf{1}_{M \times N} + \delta I)^{-1} \mathbf{1}_{N \times M} \mathbf{1}_{M \times 1} = 1 - \mathbf{1}_{1 \times N} (\mathbf{1}_{N \times N} + \frac{\delta}{M})^{-1} \mathbf{1}_{N \times 1}, \tag{63}
$$

which is definitely finite. Hence, [\(61\)](#page-8-4) yields that  $\mathcal{E}_i(h) \to 0$  in the limit as  $h \to 0^+$ . In the limit as  $\delta \to 0$  or  $M \to \infty$ , we see that  $A_0(\delta, M) \to 0$ , and our second claim holds. By similar arguments we may show that the same result holds for every  $t \in [t_{\ell-1}, t_{\ell}]$ , and not just on the collocation points (consistency in the uniform norm).  $\Box$ 

#### 3.2 Linear stability analysis of a system of ODEs.

We will now extend our stability analysis to the case of a system of ODEs of the form [\(7\)](#page-2-3). We will first provide a partial result for the multicollocation scheme for the case where  $A$  is diagonalizable (Theorem [3.3\)](#page-8-5), that will be of interest in its own right for the extension of our results to the PDE case.

**Proposition 3.3.** Consider the ODE system [\(7\)](#page-2-3) and let  $\{\lambda_i, \dots, i = 1, \dots, d\}$  be the eigenvalues of A. Then,

- (i) The PINN multicollocation scheme (as in Theorem [3.1\)](#page-5-3) is asymptotically stable if  $\max_i \lambda_i h \to 0^-$  and unstable if  $\lambda_i h \to 0^+$  for at least one i. Also for  $\max_i \lambda_i h \to -\infty$ , the scheme is asymptotically stable.
- (ii) The single collocation scheme (as in Proposition [3.2\)](#page-6-2) is A-stable a.s. if  $z < 0$  and  $\max_i \lambda_i < 0$ .

*Proof.* By the assumption of diagonalizability of A there exists a similarity transformation S such that  $A = S^{-1}DS$ where  $D = diag(\lambda_1, \dots, \lambda_d)$ , with  $\lambda_i$ , the eigenvalues of the matrix A. Upon defining  $\boldsymbol{z} = S \boldsymbol{u}$ , we can transform system [\(7\)](#page-2-3) to

<span id="page-9-0"></span>
$$
\frac{dz}{dt} = Dz, \quad z(0) = z_0 := S u_0. \tag{64}
$$

System [\(64\)](#page-9-0) is a decoupled system of ODEs of the form

$$
\frac{dz_i}{dt} = \lambda_i z_i, \quad i = 1, \cdots, d \tag{65}
$$

If  $\lambda_i \leq 0$  for all i, the stability of the scheme is dominated by the eigenvalue with minimum value of  $|\lambda_i|$ . Let  $|\lambda^*| = \min_{i=1,\dots,d} |\lambda_i|$ . Then applying Theorem [3.1](#page-5-3) and Proposition [3.2](#page-6-2) we obtain the stated result. □

For the non-diagonalizable case, we state the following Theorem.

**Theorem 3.3.** For a general non-diagonalizable linear system of ODEs of the  $(7)$ , where A is a matrix with its eigenvalues all real and negative, the single collocation PINN scheme is asymptotically stable.

*Proof.* Suppose that the matrix A is not diagonizable because its spectrum consists of non-single eigenvalues. In this case the matrix A can be transformed in Jordan block form using the Jordan decomposition  $A = PJP^{-1}$ ,

$$
J = \begin{pmatrix} J_1 & 0 & & \\ 0 & J_2 & & \\ & \ddots & & \\ 0 & & J_q \end{pmatrix}, \quad J_m = \begin{pmatrix} \lambda & 1 & 0 & & \\ 0 & \lambda & 1 & 0 & \\ & & \ddots & & \\ & & & \lambda & 1 & \\ & & & & 0 & \lambda \end{pmatrix} \in \mathbb{R}^{m \times m}
$$
 (66)

where each  $J_{\ell}$  is a Jordan block of dimension  $\ell$ , corresponding to the multiplicity of the corresponding eigenvalue.

The invertible matrix P, whose columns are generalized eigenvectors of A, defines a new coordinate system  $u = Pz$ under which we can transform the system [\(7\)](#page-2-3) to

$$
\frac{dz}{dt} = Jz, \quad z(0) = z_0 := P^{-1}u_0.
$$
\n(67)

For these systems it suffices to study the stability analysis on the single Jordan block [\[18\]](#page-17-23). Hence, we now consider the ODE for each Jordan block  $J_m$ ,

<span id="page-9-2"></span>
$$
\frac{dy}{dt} = J_m y. \tag{68}
$$

Setting  $\mathbf{y} = (y_1, \dots, y_m)^T$ , we now assume the following expansions for  $y_\ell, \ell = 1, \dots, m$ ,

<span id="page-9-1"></span>
$$
y_{\ell}(t) = y_{\ell}(t_{i-1}) + (t - t_{i-1}) \sum_{j=1}^{M} w_{j,\ell}^{(i)} e^{-a_{j,\ell}^{(i)}(t - \tau_{j,\ell}^{(i)})^2}, \ \ t \in [t_{i-1}, t_i],
$$
\n
$$
(69)
$$

allowing, if necessary, different parameters for each  $\ell$ .

Upon differentiation of the above expressions we obtain the errors  $\mathcal{E}_{\ell}^{(i)}$  $\mathcal{L}_{\ell}^{(i)} := \frac{dy_{\ell}}{dt}(t_i) - \lambda y_{\ell}(t_i) - y_{\ell+1}(t_i)$  that for each collocation point  $t_i$ , in terms of the following vectors

$$
q_{\ell}^{(i)} = e^{-a_{j,\ell}^{(i)}(t - t_{j,m}^{(i)})^2}, \quad j = 1, \cdots, M \in \mathbb{R}^M,
$$
  
\n
$$
k_{\ell}^{(i)} = (1 - \lambda h - 2a_{j,\ell}^{(i)}h(t_i - \tau_{j,\ell}^{(i)}))e^{-a_{j,\ell}^{(i)}(t - \tau_{j,\ell}^{(i)})^2}, \quad j = 1, \cdots, M \in \mathbb{R}^M,
$$
  
\n
$$
w_{\ell}^{(i)} = w_{j,\ell}^{(i)}, \quad j = 1, \cdots, M,
$$
\n(70)

(where for simplicity we have dropped the explicit  $i$  dependence) as

<span id="page-10-2"></span>
$$
\mathcal{E}_{\ell}^{(i)} = \langle w_{\ell}^{(i)}, k_{\ell}^{(i)} \rangle - h \langle w_{\ell+1}^{(i)}, q_{\ell+1}^{(i)} \rangle - \alpha_{\ell}^{(i)}, \quad \ell = 1, \cdots, m-1, \quad \mathcal{E}_{m}^{(i)} = \langle w_{\ell}^{(i)}, k_{m}^{(i)} \rangle - \alpha_{m}^{(i)},
$$
\n
$$
\alpha_{m}^{(i)} = \lambda y_{m,i-1}, \quad \alpha_{\ell}^{(i)} = \lambda y_{\ell,i-1} + y_{\ell+1,i-1}.
$$
\n(71)

The choice of the (random) weights  $w_{\ell}^{(i)}$  $\ell^{(i)}$ ,  $\ell = 1, \cdots, m$ , will be made so that the square error at the collocation points  $\{t_i\}$  is minimized, i.e.

$$
(w_{\ell}^{(i)}, \ell = 1, \cdots, m) \in \arg\min_{(w_{\ell}^{(i)}, \ell = 1, \cdots, m)} \sum_{\ell=1}^{m} (\mathcal{E}_{\ell}^{(i)})^2
$$
 (72)

The first order conditions reduce to  $\mathcal{E}_{\ell} = 0$ , for  $\ell = 1, \cdots, m$ , which in turn reduces to

<span id="page-10-0"></span>
$$
\langle w_m^{(i)}, k_m^{(i)} \rangle = \alpha_m^{(i)} =: r_m^{(i)},
$$
  
\n
$$
\langle w_\ell^{(i)}, k_\ell^{(i)} \rangle = \underbrace{h \langle w_{\ell+1}^{(i)}, q_{\ell+1}^{(i)} \rangle + \alpha_\ell^{(i)}}_{:=r_\ell^{(i)}}, \ \ell = m - 1, \cdots, 1
$$
\n(73)

Note that system [\(73\)](#page-10-0) can be solved in terms of a backward iteration scheme, starting from the equation for  $k_m^{(i)}$ , then proceeding to  $\ell = m-1$  and substituting the solution obtained for  $w_m^{(i)}$  to get  $w_{m-1}^{(i)}$ , and working similarly backwards up to  $\ell = 1$ .

In the same fashion as for the diagonal case we have that  $(73)$  yields

<span id="page-10-1"></span>
$$
w_{\ell}^{(i)} = \frac{r_{\ell}^{(i)}}{\|k_{\ell}^{(i)}\|^2} k_{\ell}^{(i)}, \quad \ell = m, \cdots, 1,
$$
\n(74)

which is a solution in an implicit form since the the coefficients  $r_{\ell}^{(i)}$  $\mathcal{L}_{\ell}^{(i)}$ , given by the RHS of [\(73\)](#page-10-0), depend on  $w_{\ell+1}^{(i)}$ . Note however, that the definition of  $r_{\ell}^{(i)}$  $\ell_{\ell}^{(i)}$  can be interpreted as a backward iteration scheme, the solution of which will turn [\(74\)](#page-10-1) into an explicit solution for the weights.

Substituting [\(74\)](#page-10-1) into the definition of  $r_{\ell}^{(i)}$  $\ell^{(i)}$ , yields the backward iteration scheme

$$
r_m^{(i)} = \alpha_m^{(i)},
$$
  
\n
$$
r_\ell^{(i)} = \Lambda_{\ell+1} r_{\ell+1} + \alpha_\ell^{(i)}, \quad \ell = m-1, \cdots, 1
$$
  
\n
$$
\Lambda_{\ell+1}^{(i)} = h \frac{\langle k_{\ell+1}^{(i)}, q_{\ell+1}^{(i)} \rangle}{\|k_{\ell+1}^{(i)}\|^2},
$$
\n(75)

that can be explicitly solved in terms of  $\Lambda_{\ell}^{(i)}$  $\stackrel{(i)}{\ell},\,\alpha_\ell^{(i)}$  $\ell^{(1)}$  and the combined with [\(74\)](#page-10-1) to obtain the required solution for the weights.

By induction and recalling the dependence of  $a$  on  $y$  (see [\(71\)](#page-10-2) we obtain the general formula

<span id="page-10-3"></span>
$$
r_{m-\nu}^{(i)} = \lambda y_{m-\nu,i-1} + \sum_{\ell=0}^{\nu-1} \left( \prod_{\sigma=\ell+1}^{\nu-1} \Lambda_{m-\sigma}^{(i)} \right) (1 + \lambda \Lambda_{m-\ell}) y_{m-\ell,i-1}, \quad \nu = 1, \cdots, m-1,
$$
 (76)

where the convention that  $\prod_{s=1}^{0} \Lambda_s^{(i)} := 1$  is used.

We are now in position to reconstruct the weights  $w_{\ell}^{(i)}$  $\mathcal{U}_{\ell}^{(i)}$ , using [\(74\)](#page-10-1) and [\(76\)](#page-10-3), to obtain an expression for  $y_i =$  $(y_{1,i}, y_{2,i}, \dots, y_{m,i})^T$  in terms of  $y_{i-1} = (y_{1,i-1}, y_{2,i-1}, \dots, y_{m,i-1})^T$  using the representation [\(69\)](#page-9-1).

After some algebra we see that

$$
\mathbf{y}_i = M^{(i)} \mathbf{y}_{i-1},\tag{77}
$$

where  $M^{(i)}$  is the upper triangular  $m \times m$  matrix

$$
M^{(i)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & (1 + \lambda \Lambda_{m-2}^{(i)}) & \Lambda_{m-2}^{(i)} (1 + \lambda \Lambda_{m-1}^{(i)}) & \Lambda_{m-2}^{(i)} \Lambda_{m-1}^{(i)} (1 + \lambda \Lambda_m^{(i)}) \\ 0 & (1 + \lambda \Lambda_{m-1}^{(i)}) & \Lambda_{m-1}^{(i)} (1 + \lambda \Lambda_m^{(i)}) \\ 0 & 0 & (1 + \lambda \Lambda_m^{(i)}) \end{pmatrix}
$$
(78)

The eigenvalues of the matrix  $M_i$  are the diagonal elements  $M_{jj}^{(i)} = 1 + \lambda \Lambda_j^{(i)}$ ,  $j = 1, \dots, m$ . Note that as of Prop. [3.2,](#page-6-2) we have that  $|1 + \lambda \Lambda_j^{(i)}| < 1$ .

Recursively, we get:

$$
\mathbf{y}_i = M^{(i)} M^{(i-1)} \cdots M^{(1)} \mathbf{y}_0 \tag{79}
$$

As each  $M^{(j)}$ ,  $j = 1, \ldots, i$  is an upper triangular matrix the transition matrix  $M^{(i)}M^{(i-1)} \cdots M^{(1)}$  is also an upper triangular matrix with eigenvalues the product of the eigenvalues of each  $M^{(j)}$ ,  $j = 1, \ldots, i$ .

Hence, the transition matrix  $M^{(i)}M^{(i-1)} \cdots M^{(1)}$  has m distinct eigenvalues and therefore the m corresponding eigenvectors, say,  $\{v_1^{(i)}, v_2^{(i)}, \dots v_m^{(i)}\}$  form a basis in  $\mathbb{R}^m$ . Hence, we can write  $y_0$  as:

$$
\mathbf{y}_0 = \sum_{j=1}^m c_j^{(i)} \mathbf{v}_j^{(i)}.
$$
 (80)

Since:

$$
M^{(i)}M^{(i-1)}\cdots M^{(1)}\mathbf{v}_j^{(i)} = \prod_{p=1}^i (1+\lambda\Lambda_j)^p \mathbf{v}_j^{(i)}, \ j=1,2,\ldots,m,
$$
\n(81)

we have:

$$
\mathbf{y}_{i} = c_{1}^{(i)} M^{(i)} M^{(i-1)} \cdots M^{(1)} \mathbf{v}_{1}^{(i)} + c_{2} M^{(i)} M^{(i-1)} \cdots M^{(1)} \mathbf{v}_{2}^{(i)} + \cdots + c_{m}^{(i)} M^{(i)} M^{(i-1)} \cdots M^{(1)} \mathbf{v}_{m}^{(i)}, \tag{82}
$$

or

$$
\mathbf{y}_{i} = c_{1}^{(i)} \prod_{p=1}^{i} (1 + \lambda \Lambda_{1})^{p} \mathbf{v}_{1}^{(i)} + c_{2}^{(i)} \prod_{p=1}^{i} (1 + \lambda \Lambda_{2})^{p} \mathbf{v}_{2}^{(i)} + \dots + c_{m}^{(i)} \prod_{p=1}^{i} (1 + \lambda \Lambda_{m})^{p} \mathbf{v}_{m}^{(i)}.
$$
 (83)

But each product  $\prod_{p=1}^{i}(1+\lambda\Lambda_j)^p\to 0, j=1,2,\ldots,m$ , as  $i\to\infty$  because  $|(1+\lambda\Lambda_j)^p|< 1$   $\forall p,j.$  Consequently:

$$
\lim_{i \to \infty} \mathbf{y}_i = \mathbf{0},\tag{84}
$$

 $\Box$ 

and therefore because of  $u = Pz$ , and, [\(68\)](#page-9-2), we have that the PINN scheme is asymptotically stable.

Remark 2. Note that the above proof can be extended to every matrix A over the field of complex numbers, i.e., for matrices also with multiple complex eigenvalues (see for example in [\[54\]](#page-18-21)).

Remark 3. The results of this section can be applied also in the case of linear parabolic equations, after discretization in space and time using standard methods, such as finite differences schemes, or using an eigenfunction expansion, in which case one can apply Proposition [3.3](#page-8-5) (see example [4.3](#page-13-0) below).

# 4 Numerical results

In this section, we present the numerical approximation accuracy and computational costs of the proposed PINN scheme. We focus on assessing the accuracy of an approximated solution  $\hat{y}(t)$ , of the true solution  $y(t)$  measured in terms of the  $L^2$ -error metric. To evaluate the performance of the PINN scheme, we utilize a set of benchmark linear ODE problems. These include the stiff Prothero-Robinson equation, a 2D linear ordinary differential equation showcasing periodic behavior around a central point, and a linear diffusion-reaction PDE discretized using finite differences.

Furthermore, we compare the performance of the proposed scheme in terms of both numerical approximation accuracy and computational cost versus various traditional implicit schemes. In particular, the implicit traditional schemes employed are the Backward Euler, implicit midpoint, implicit trapezoidal (Crank-Nikolson), the 2-stage Gauss scheme and the 2 and 3 stages Radau schemes. For our illustrations, we employed 3-stages (M=3, N=9) and 10 stages (M=10 and N=30) schemes. The upper bounds  $a_U$  are computed accordingly to the requirements proved in the previous sections. Also to be computationally efficient we randomize only one time, once and for all, the internal weights of the PINN, and we keep them fixed along the entire time domain. In this way, we can precompute the pseudoinverse matrix for the solution of the problem. In particular for the pseudo inverse we compute a Complete Orthogonal Decomposition (COD) (or known as double-sided QR decomposition with regularization) that in our experiments gave better results than the classical SVD decomposition, resulting in better numerical errors (for details about comparison of SVD and COD for PINNs see [\[12\]](#page-16-16)). All the computations are carried on single core of a Intel Core i7-10750H CPU 2.60GHz, with 16GB of RAM running Matlab 2020b.

#### 4.1 Example 1: Prothero-Robinson stiff ODE

Here, we solve the Prothero-Robinson stiff problem:

$$
\frac{dy}{dt} = \lambda(y - \phi(t)) + \phi'(t), \quad y(0) = \phi(0).
$$
\n(85)

The analytical solution is given by  $\phi(t)$ , (here  $\phi(t) = \sin(t)$ ) but for large negative value of  $\lambda$ , (here  $\lambda = -1000$ ), the problem becomes stiff.

The above linear non-homogeneous stiff ODE can be viewed as a linear homogeneous ODE with an additional forcing term.



<span id="page-12-0"></span>Figure 2: Numerical approximation accuracy and computational cost for the stiff Prothero-Robinson ODE benchmark problem in [0, 10 $\pi$ ]. The PINN method, with random projections, uses  $M = 3$  and  $M = 10$  collocation points in each subinterval of size h and  $N = 9$  and  $N = 30$  neurons, respectively. The numerical approximation accuracy of the PINNs are reported in dashed lines, while implicit RK schemes with solid lines. (a)  $L^2$ -error in terms of fixed time step h. (b)  $L^2$ -error with respect to machine/computational execution time.

We solve the problem in the interval  $[0, 10\pi]$  using various fixed time steps, h. Figure [2](#page-12-0) compares the convergence of our proposed PINNs method, leveraging random projections, with several implicit RK solvers. The PINNs method, employing both 3 and 10 collocation/stage schemes, consistently outperforms the 2-stage Gauss and 3-stage Radau RK solvers in terms of accuracy for time steps ranging from  $h = 1E-3$  to  $h = 10$ . Moreover, both PINNs and implicit RK methods exhibit comparable computational costs, as shown in panel (b) of Figure [2.](#page-12-0) Notably, the PINNs schemes demonstrate remarkable accuracy even with relatively large time steps. For instance, they achieve an  $L^2$ -error of 1E−04 with a time step  $h = 10$  and 1E−06 with  $h = 1$ . Furthermore, PINNs can attain an  $L^2$ –error approaching machine precision (approximately 1E−12) using a comparatively large time step of  $h = 0.01$ .

#### 4.2 2D Linear system of a harmonic oscillator

To assess the performance of numerical schemes when the eigenvalues lie along the imaginary axis, we consider a simple 2D linear system of a harmonic oscillator:

<span id="page-13-2"></span>
$$
\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \mathbf{y}(t), \qquad \mathbf{y}(0) = (1,0) := \mathbf{y}_0. \tag{86}
$$

The analytical solution, given by  $y(t) = (y_1(t), y_2(t)) = (\cos(\omega t), -\omega \sin(\omega t))$ , describes a periodic orbit on the ellipse

$$
\mathcal{E}_{\omega} \equiv \{ (y_1, y_2) \in \mathbb{R}^2 \, : \, \omega^2 y_1^2 + y_2^2 = \omega^2 \}
$$

and does not converge to a stationary point. However, an A-stable method, can bear the danger of turning an unstable or a non-asymptotically stable IVP into an asymptotically stable discrete-time system. This is the case of implicit backward Euler method, for which the solution does not stay on the ellipse  $\mathcal{E}_{\omega}$  approaching zero for long times, as depicted in Figure [3\(](#page-13-1)a). While in contrast, for other schemes, as the Explicit Forward Euler, the solution may drift away as depicted in Figure [3\(](#page-13-1)b).



<span id="page-13-1"></span>Figure 3: Stability around the imaginary axis. Simple 2D linear system in [\(86\)](#page-13-2), with  $\omega = 10$ , for  $t \in [0, 2\pi]$ . step size is fixed at  $h = 0.01$ . (a) Explicit Euler, (b) Implicit Euler, (c) Trapezoidal, (d) PIRPNN with  $M = 3$ , (e) PIRPNN with  $M = 10$ .

For our experiments, we set  $\omega = 10$  and the domain of integration  $[0, 2\pi]$ . In Figure [3](#page-13-1) we depict the behavior of the solutions for a fixed  $h = 0.01$  for different methods. As shown explicit and implicit Euler methods do not provide the right stability, while the two PINNs (with  $M = 3$  and  $M = 10$  collocations) provide the same *correct* stability of the trapezoidal scheme, remaining on the ellipse for the entire time horizon.

We also solved the problem, for various fixed values of the time step  $h$ . The numerical results of the proposed PINNs, leveraging random projections, and the other traditional implicit RK schemes are depicted in Figure [4.](#page-14-0) The PINNs method with 3 collocations achieves comparable accuracy to the 2-stage Gauss RK scheme. In contrast, the PINNs with 10 collocations rapidly converges to an  $L^2$ –error of 1E–08 for large time steps but plateaus at this level for small time steps, potentially due to numerical limitations in the matrix pseudo-inversion and/or accumulation of errors on the boundary of the stability region. Notably, the PINNs method, employing 10 collocation/stage schemes, consistently outperforms the 2-stage Gauss scheme in terms of accuracy for time steps ranging from  $h = 1E-03$  to  $h = 1$ . The Radau-3 scheme is the only method that surpasses the convergence of both PINNs, but only for time steps smaller than 1E−02.

#### <span id="page-13-0"></span>4.3 A linear diffusion-reaction PDE problem

We consider a simple, linear diffusion-reaction PDE given by:

<span id="page-13-3"></span>
$$
u_t = \nu u_{xx} - \lambda u. \tag{87}
$$

with Neumann BCs in  $[0, \pi]$  and initial conditions  $u(x, 0) = a \cos(2x) + c$  (here  $a = 0.4, c = 1.5$ ). Based on the above, the analytical solution  $u(t, x)$  of Eq. [\(87\)](#page-13-3) reads:

$$
u(t, x) = a \exp(-(4\nu + \lambda)t)\cos(2x) + c \exp(-\lambda t). \tag{88}
$$



<span id="page-14-0"></span>Figure 4: Performance for the 2d linear ODE harmonic oscillator in Eq [\(86\)](#page-13-2), with  $\omega = 10$  in the time interval [0,  $2\pi$ ]. For the PINNs we use  $M = 3$  and  $M = 10$  collocation points in each subinterval of size h and  $N = 9$  and  $N = 30$ neurons, respectively. Results with PINNs are reported in dashed lines, while the ones using implicit RK schemes with solid lines. (a)-(c)  $\dot{L}^2$ -error in terms of fixed time step h, for  $y_1$  and  $y_2$  respectively. (b)-(d)  $\dot{L}^2$ -error with respect to machine/computational execution time, for  $y_1$  and  $y_2$  respectively.

For our simulations, we have set  $\nu = 0.001$  and  $\lambda = 10$ . For the discretization in space of the PDE, we employed a second order centered finite difference scheme over a grid of  $n + 2$  points  $x_i$ ,  $i = 0, 1, \ldots, n, n + 1$ . We select in particular both  $n = 100$  and  $n = 200$  leading to 100 and 200 ODEs, respectively. The boundary conditions are hardwired into the equations. The resulting system of  $n$  ODEs is given:

$$
\frac{du(x_i, t)}{dt} = \frac{du_i}{dt} = \nu \frac{u_{i+1} - 2u_i - u_{i-1}}{\Delta x^2} - \lambda u_i, \qquad i = 1, ..., n
$$
\n
$$
u_0 = \frac{4u_1 - u_2}{3\Delta x}, \qquad u_{n+1} = \frac{4u_n - u_{n-1}}{3\Delta x},
$$
\n(89)

<span id="page-14-1"></span>where  $\Delta x = 2\pi/(n+1)$ . The resulting ODE [\(89\)](#page-14-1) is of the general form [\(9\)](#page-3-1), with the matrix A corresponding to the finite difference approximation matrix for the 1-D Laplacian. The resulting system presents stiffness properties on account of the spectrum of the matrix A. The numerical results, in the time interval  $[0, 1]$  using various fixed time steps h, of the proposed PINNs, leveraging random projections, and the other traditional immplicit RK schemes are depicted in Figure [5,](#page-15-0) with the first and second rows corresponding to  $n = 100$  and  $n = 200$  discretization points, respectively. Also in this case the 3 collocation PINN scheme show comparable result with the 2-stage Gauss RK scheme. The PINNs method, 10 collocation/stage schemes, consistently outperforms all the other implicit RK solvers in terms of accuracy for time steps ranging from  $h = 1E-03$  to  $h = 1E-01$ . While PINNs methods generally incur slightly higher computational costs due to the use of  $N = 3M$  neurons, as illustrated in panel (b) of Figure [5,](#page-15-0) this overhead become more evident when handling high dimensional system (100 or 200 ODEs). Reducing PINNs overparametrization, as suggested in [\[15\]](#page-16-9), could enhance computational efficiency. Notably, the finite difference discretization limits the



<span id="page-15-0"></span>Figure 5: The linear Diffusion-Reaction PDE in Eq [\(87\)](#page-13-3), with  $\nu = 0.001$  and  $\lambda = 10$  in the time interval [0, 1]. For the PINNs we used  $M = 3$  and  $M = 10$  collocation points in each sub-interval of size h and  $N = 9$  and  $N = 30$ neurons, respectively. The results with the PINNs are reported in dashed lines, while implicit RK schemes with solid lines. (a)  $L^2$ –error in terms of fixed time step h. (b)  $L^2$ –error with respect to machine/computational execution time.

achievable accuracy for all methods, with a saturation around 1E-8 for  $n = 100$  and only slightly higher for  $n = 200$ . The 10-collocation PINNs method approaches this limit using a relatively large time step of 1E−01.

#### 5 Conclusion

We presented a stability analysis of linear (stiff) ODEs for physics-informed neural networks (PINNs) with random projections using radial basis functions (RBFs) as activation functions. We proved that such PINNs, with appropriate sampling of the hyperparameters of the RBFs, are consistent schemes and unconditionally stable for stiff systems for all step sizes and that they have the correct stability in the complex plane. We also proved the asymptotic stability of the scheme, for the general case of linear systems of ODEs, and demonstrated the extension of the analysis for a linear parabolic PDE. This is the first time that such a proof is given. Numerical simulations to various benchmark problems are also provided. We also compared the computational implementation cost, convergence, numerical approximation accuracy of the proposed PINNs for various step sizes with traditional implicit Runge-Kutta schemes. We showed that the proposed scheme outperforms the other implicit schemes in terms of numerical approximation accuracy, while having a comparable computational cost, for a wide range of step sizes. We believe that our work will open the path for a more rigorous numerical analysis of scientific machine learning algorithms for the solution of both the forward and inverse problems for differential equations.

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