Order from chaos

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At first glance, the words, "chaos" and "order" seem to be contradictory. Indeed they are contradictory in terms of common English definitions of the words, but not in terms of their mathematical usages. The English word definition of the word chaos is, "great disorder or confusion," [1]. Quite to the contrary, the mathematical notion of chaos explains that data which seems to be randomly generated may have actually been generated by a deterministic and ordered process. In this sense, the field might have profitably been called, "order theory," instead of "chaos theory." There are two common usages of the phrase, "order from chaos." The first refers to looking for deterministic "chaotic" models to explain complex phenomenon, thus "order in chaos." The second furthermore refers to the fact that some chaotic dynamical systems can also exhibit regular islands of simplicity within their phase space, or "order from chaos." We will discuss each in turn.

1 Order in chaos

That seemingly simple and deterministic evolution rules can give rise to extremely complicated motion dates to Henri Poincare' [2], in the setting of celestial mechanics. Perhaps the most famous example of unpredictable behavior is due to E. Lorenz [3], who discovered that even the most simplified models of the weather could produce data for which it is impossible to make long-term forecasts. Lorenz identified, "the butterfly effect," or "sensitive dependence to initial conditions," whereby even small measurement errors quickly grow to swamp the signal. In 1975, T.Y. Li and J.A. Yorke published a paper entitled, "Period-three implies chaos," [4]. The most famous impact of Yorke's mathematical paper was that it coined the phrase, "Chaos," the field of study describing the nonlinear effect in which even the simple systems of Poincare, Lorenz and others can display sensitive dependence in a bounded domain. Chaos theory has attracted a great deal of popular attention, partly in fact due to its sexy title. Partly the appeal is also due to scientific tradition to search for simple mechanistic explanation, since the time of Newton; see Blake's rendition of Newton in Fig. 2.

Put simply, the "phase space" could be described as the set where the collection of all relevant variables live, which create a closed description of the time evolution of the system. For example, for a single pendulum, we need angular position and angular momentum, to uniquely define all solutions. For an LRC circuit, we require charge and current through the simple wire loop. The dimension of the systems is the dimension of the phase space. A billiards table's worth of balls, for example, requires many variables to completely describe, two variables for position and two for momentum for each of the many balls, for this example of a high dimensional system,

Chaos Theory has been a highly interdisciplinary and successful field in finding deterministic chaos, or order, in what was once thought must be just noise, or at least extremely high-dimensional effects, with numerous explicit examples in areas ranging from biology , to electronics engineering, to chemistry, to celestial mechanics, to brain and heart physiology, etc. Thus, an old paradigm that extremely complicated datum must be due to extremely complicated effects is false. See Fig. 1. One popular misuse of these

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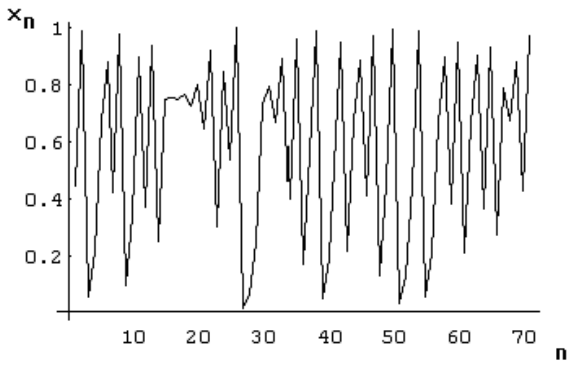

Figure 1: This seemingly stochastic data was actually generated by a "simple," low-dimensional and deterministic process. This time-series plot of the state x_n versus time n was actually generated by the simple and deterministic dynamical system $x_{n+1} = ax_n(1-x_n)$ called the logistic map. ($a = 4.0$, $x_0 = 0.44$).

ideas is a belief that all complicated datum must have underlying order in chaos. Asked plainly: is what seems complicated always really just simple? The answer is, of course, no; it is still true that noise and also otherwise high-dimensional effects can also be responsible for complexity. What is true is that what seems complicated may *sometimes* be simple, in the sense of having a low-dimensional chaotic model. To cite one popular "million-dollar" question, the stock market prices constitute unarguably extremely complicated data, but is there underlying order in chaos here, or is the explanation due to intractably high-dimensionalality?

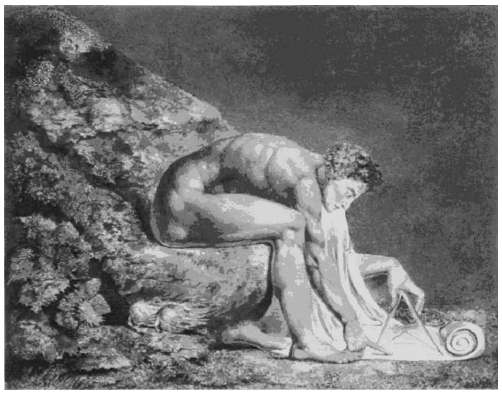

Figure 2: Blake's "Newton," 1795. To Blake, Newton was the archrepresentative of material philosophy. Norton Anthology II. P. 79 (n. 6). Blake's Newton has been related to critical and historical categories of order and disorder.

2 Order from chaos

Some chaotic systems have a peculiar tendency to, sometimes, exhibit simple behavior. This is often described as regular islands in an otherwise chaotic phase space. The classic mathematical example of

regular islands arises in Hamiltonian systems, which can exhibit a phase space of solutions in which both chaotic solutions are intermingled and bounded by nested islands around, islands, around islands, etc., of KAM-like tori (circle-like integrable solutions) [5, 6].

As a different usage of the phrase, "Order from chaos" and "regular islands" are also commonly used broadly to refer the presence of an attractor in what one might think should be a system which should behave in a complicated manner. For example, consider a thought experiment of the game of pool in which we assume no friction so that the balls do not slow to a stop. Following the path of one specific ball, say the 7-ball, is an extremely complicated problem displaying sensitive dependence, since a small change in the velocity or position of the ball impacts the next collision with the wall or the next ball, which multiplies in effect upon subsequent collisions. However, there are several attractors, which if within their basin, the motion becomes quite simple; these are the several pocket holes. Once within the rim of a pocket, then falling in becomes inevitable, and would require a relatively large energy perturbation to prevent. In this sense, we can say that the regions of phase space corresponding to being stuck in the billiards pocket, are regular islands in a chaotic sea.

Simply as analogy, such descriptions of islands in chaotic seas have been extended by some to explain emergence of explicit phenomenon from complicated processes. There is perhaps no more pressing question than the origins of life. Proponents of emergence theories could say that life in the original Hadean seas (the "primordial" seas on early Earth) gave rise to life through a capture process such as the billiards game. [7, 8]. While randomly choosen initial conditions in the billiards game may each individually be unlikely to lead to a putting a ball in the pocket, and certainly any analogous attractors in chemical processes must be exceedingly unlikely, the way to win an unlikely bet is to play very quickly, over-and-over, over many eons. Exactly such may have been the formula which lead to complex organic molecules, so say proponents of this theory.

Emergence and order from chaos have also been used to describe processes whose attractors are surprisingly complicated themselves, as sets, but arise from surprisingly simple rules. Michael Barnsley has called the following "The Chaos Game" [9]. First we label the vertices of a triangle, 1, 2, 3. Using a random number generator, such as a six-sided die, identify probabilities to each triangle vertex, say die sides 1-2 to vertex 1, die sides 3-4 to vertex 2, and die sides 5-6 to vertex 3. Roll the die to randomly select on of the vertices, and record its planar coordinates (x,y). Then roll the die again to randomly select another vertex, and record the point halfway between the resulting new vertex and the current (x,y) as the new (x,y). Repeat indefinitely. For each newly defined (x,y), we record a dot for a pictoral record. At first blush, most people's intuition guesses that the result will uniformly fill the triangle with a smattering of dots. The mathematical fact is however perhaps surprising. The result is an extremely intricate structure, a fractal called the Sierpinski gasket. See Fig. 3. Since this simple rule gives rise to such a complicated structure, the argument goes, perhaps many of the other intricacies we see around us might have emerged from other simple rules.

Finally, we mention the meaning of "order from chaos," as developed from theories of Nobel prize winning physicist Ilya Prigogine. Priggogine's work on dissipative structures of systems held from thermal equilibrium has been used to study self-organizing systems [10]. Prigogine has defined [10] complexity as the ability to switch between different modes of behavior as the environmental conditions vary. Prigogne has described a phenomenon in which far-from-equilibrium systems transition "from being to becoming," which some describe as order from chaos.

References

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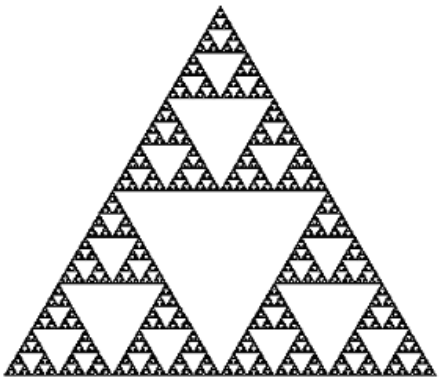

Figure 3: From such a simple algorithm as "The Chaos Game" [9] emerges the extremely intricate fractal shown, called the Sierpinski gasket.

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