Eur. Phys. J. B **57**, 481–486 (2007)
DOI: 10.1140/epjb/e2007-00186-3

**THE EUROPEAN PHYSICAL JOURNAL B**

# Decline of minorities in stubborn societies

M. Porfiri[1,a], E.M. Bollt[2], and D.J. Stilwell[3]

[1] Department of Mechanical, Aerospace and Manufacturing Engineering, Polytechnic University, Brooklyn, NY 11201, USA
[2] Department of Mathematics and Computer Science, Clarkson University, Postdam, NY 13699-5815, USA
[3] The Bradley Department of Electrical and Computer Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

Received 6 September 2006 / Received in final form 27 March 2007
Published online 29 June 2007 – 

**Abstract.** Opinion compromise models can give insight into how groups of individuals may either come to form *consensus* or clusters of opinion groups, corresponding to *parties*. We consider models where randomly selected individuals interact pairwise. If the opinions of the interacting agents are not within a certain confidence threshold, the agents retain their own point of view. Otherwise, they constructively dialogue and smooth their opinions. Persuasible agents are inclined to compromise with interacting individuals. Stubborn individuals slightly modify their opinion during the interaction. Collective states for persuasible societies include extremist minorities, which instead decline in stubborn societies. We derive a mean field approximation for the compromise model in stubborn populations. Bifurcation and clustering analysis of this model compares favorably with Monte Carlo analysis found in the literature.

**PACS.** 05.45.-a Nonlinear dynamics and chaos – 89.65.-s Social and economic systems – 89.75.-k Complex systems

## 1 Introduction

Opinion formation involves information sharing among individuals with different viewpoints. Information exchange leads to collective states, where either agents achieve consensus on a common opinion or multiple opinion clusters arise. In *compromise models* [1,2], the opinion of each individual is a continuous variable in an interval $[-W, W]$, with $W > 0$. Opinion formation is driven by binary interactions of randomly selected agents. If the difference between the opinions of the interacting agents is larger than 1, the agents refuse to constructively dialogue and they rather preserve their initial opinions. Otherwise, the agents average their initial opinion. The binary interaction is described by

$$x' = x + \mu(x_\star - x) \quad (1)$$

$$x'_\star = x_\star + \mu(x - x_\star) \quad (2)$$

where the pair $(x, x_\star)$ is the opinions of the randomly selected individuals before the meeting and $(x', x'_\star)$ their opinions after the interaction. The parameter $0 < \mu \leq 1/2$ measures the *persuasibility* of the competing agents. For small values of $\mu$, the agents slightly change their opinions during the meeting. Small values of $\mu$ refer to *stubborn* societies, where individuals are not acquiescent and refrain from compromising with competing individuals. Large values of $\mu$ imply inclination of the population to compromise. For $\mu = 1/2$ the competing agents fully compromise and after the interaction they share the same opinion. The updated opinions (1) and (2) are still in $[-W, W]$ and their distance is reduced. Equations (1) and (2) may also describe binary collisions of point masses [3,4]. In this case, $(x, x_\star)$ represent the velocities of colliding particles and $\mu$ measures the restitution coefficient. The parameter value $\mu = 1/2$ implies a completely inelastic collision. A compromise model where each agent is interacting simultaneously with more than one individual is considered in [5]. Compromise models where individuals' opinions are discrete variables are studied for example in [6].

The ratio $u = 1/(2W)$ represent the normalized threshold of the binary interaction [2,5,7]. It represents the individuals' personal interest to change their point of view [8]. When $u = 1$, randomly selected individuals always interact constructively according to (1) and (2). In this case the information exchange leads to consensus on the zero opinion [1,2]. For smaller values of $u$, the collective state may include multiple opinion clusters, *parties*.

The effect of the *persuasibility* $\mu$ on the opinion fragmentation has been analyzed in [1]. For $\mu = 1/2$ and $0.25 < u < 0.5$ a central cluster at $x = 0$ is formed along with two cluster of extreme opinions $x \approx \pm W$. The extreme opinions are called *minorities* and they represent extremist parties. As $\mu$ decreases to 0 these extremist parties gradually disappear. For small values of $\mu$ the

[a] e-mail: mporfiri@poly.edu

opinion dynamics is slow. More individuals have the chance of meeting other individuals, of participating to the evolution of their final decision, and of being influenced by their behavior. Segregation into extremist parties is not feasible.

Mean field continuum models have been proposed to analyze opinion formation [9–13]. Time evolution of continuum models is generally described by partial integro-differential equations. In some cases, exact solutions may be found. Moreover, numerical integration of continuum models is more efficient than Monte Carlo simulations on discrete models especially when large populations are examined. Nucleation and annihilation of clusters are also better discerned. In [9,10], a mean field approximation of the compromise model for the special case $\mu = 1/2$ is considered. In [12], the limit case of $u \to 0$ is considered. In [13] mean fields of more general interaction schemes than (1) and (2) are examined, but the effect of the finite thresholds $u$ on the opinion formation is excluded. In [11] a mean field approximation of the model of [5], where each agent talks to any other agent, is presented.

We propose a mean field approximation of compromise models for the asymptotic case of stubborn population $\mu \to 0$. The master equation is similar to the Fokker-Planck equation. Numerical integration of the model yields opinion profiles in very good agreement with those found by Monte Carlo simulations in [1]. Bifurcations and clustering of the new mean field models are sensibly different from those found in [9,10]. In particular, the extremist parties predicted by [9,10] do not rise in stubborn populations.

## 2 Mean field approximation

For the limit of large populations, we replace the discrete opinion profile with the probability density function $P(x,t)$ where $t$ is a continuous time variable. For small $\Delta x$, the quantity $P(x,t)\Delta x$ measures the probability that at time $t$ the population has an opinion between $x$ and $x + \Delta x$. Loosely speaking $P(x,t)\Delta x$ represents the fraction of the entire population with an opinion between $x$ and $x + \Delta x$.

Following [12], the time rate of change of the probability function $P$ at $x \in \mathbb{R}$ may be written as the sum of two contributions

$$\frac{\partial P(x,t)}{\partial t} = I^-(x,t) + I^+(x,t). \tag{3}$$

The contribution $I^-$ is negative and represents the probability that an agent of opinion $x$ interacts at time $t$ with any other agent and thereby modifies its opinion

$$I^-(x,t) = -P(x,t) \int_{|x-y|<1} P(y,t)dy. \tag{4}$$

The other contribution $I^+$ is positive and expresses the probability that some agent of opinion different from $x$, by talking with some other agent at time $t$, updates its opinion to $x$

$$I^+(x,t) = \int_{|y_1-y_2|<1} \int_{|y_1-y_2|<1} P(y_1,t) P\left(\frac{x-(1-\mu)y_2}{\mu}, t\right) dy_1 dy_2. \tag{5}$$

By substituting (4) and (5) into (3), and through standard manipulations we obtain

$$\frac{\partial P(x,t)}{\partial t} = -P(x,t)\int_{-1}^{1} P(x+y,t)dy + \frac{1}{1-\mu}\int_{-(1-\mu)}^{(1-\mu)} P(x+y,t) P\left(x - \frac{\mu}{1-\mu}y, t\right) dy. \tag{6}$$

The initial condition is $P(x,0) = 1/(2W)$ for $x \in [-W, W]$ and zero elsewhere. The asymptotic cases of $W \to 0$ and $W \to \infty$ have been studied in [3,4] and [12] respectively. The case where $\mu = 1/2$ has been thoroughly considered in [9,10]. For an infinite number of agents, the master equation (6) is exact. For a finite number of agents the master equation approach is not exact, but it becomes exact as the number of agents diverges.

We rescale the time variable $t$ by considering the slow time scale $\tau = t/\mu$. We use the same notation for the probability function on scales $t$ and $\tau$. We then take the limit of (6) as $\mu \to 0$

$$\begin{aligned}\frac{\partial P(x,\tau)}{\partial \tau} = {}& P(x,\tau)\int_{-1}^{1} P(x+y,\tau)dy \\ & - \frac{\partial P(x,\tau)}{\partial x}\int_{-1}^{1} yP(x+y,\tau)dy \\ & - P(x,\tau)(P(-1+x,\tau) + P(1+x,\tau)).\end{aligned} \tag{7}$$

Equation (7) may be rewritten as a conservation law

$$\frac{\partial P(x,\tau)}{\partial \tau} = -\frac{\partial}{\partial x}[P(x,\tau)(h * P)(x,\tau)] \tag{8}$$

where for any positive $s$ we define

$$\rho(x,s) = \begin{cases} 1 & \text{if} \quad x \in (-s,s) \\ 0 & \text{otherwise} \end{cases}$$

and $h(x) = x\rho(x,1)$. The quantity $h * P$ may be explicitly calculated

$$(h*P)(x,\tau) = \int_{x-1}^{x+1} yP(y,\tau)dy - x\int_{x-1}^{x+1} P(y,\tau)dy. \tag{9}$$

The first integral in (9) represents the expected value of the opinion profile experienced by an agent whose opinion is $x$. We recall that in the compromise model an agent of opinion $x$ is willing to dialogue only with agents whose opinions are in $[x-1, x+1]$. The second integral in (9) represents the mass of the opinion profile experienced by an agent whose opinion is $x$.

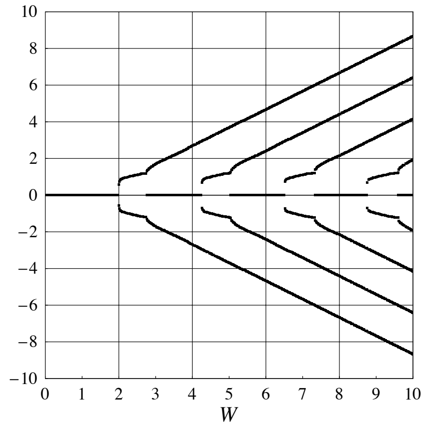


**Fig. 1.** Location of the final clusters versus the initial opinion range $W$ for stubborn societies.

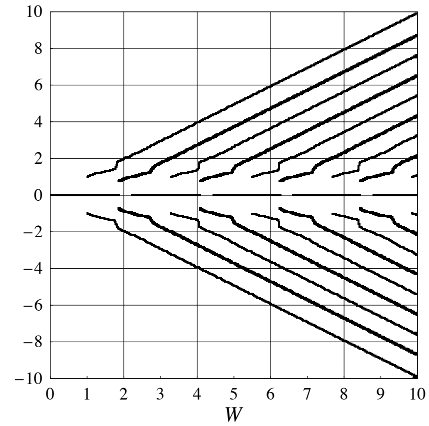


**Fig. 2.** Location of the final clusters versus the initial opinion range $W$ for acquiescent societies [9,10].

For $W \to \infty$, (8) is a Fokker-Planck equation with only the drift term; see for example [14]. Equation (8) is a conservation law since the total mass of the system is constant over time, and equals 1. Furthermore, any solution of equation (8), with initial condition $P(x,0) = \frac{1}{2W}\rho(x,W)$, remains an even function of $x$ as $\tau$ increases.

Additional properties of the solution of equation (8) could be derived following arguments similar to those presented in [13].

For $W \leq 1/2$, a solution of (8) is

$$P(x,\tau) = \rho(x, W\exp(-\tau)). \tag{10}$$

This is easily verified by noticing that for $W \leq 1/2$ and $P(x,\tau)$ in (10)

$$P(x,\tau)(h * P)(x,\tau) = x\rho(x, W\exp(-\tau)).$$

This closed-form solution yields consensus to the zero opinion as time goes to infinity. The asymptotic approach is toward a Dirac Delta distribution centered at $x = 0$, that is $P_\infty(x) = \delta(x)$. The variance of the opinion profile decays exponentially as $\sigma^2(t) = \sigma^2(0)\exp(-2\tau)$. Similar analysis for persuasible societies has been done in [9,10].

By numerical simulation we show that as $W$ increases consensus my be lost. The final distribution consists of a series of clusters whose inter-distance is larger than 1

$$P_\infty(x) = \sum_{j=1}^{\gamma} m_j \delta(x - x_j)$$

where $\gamma$ is the number of different clusters and $x_j$ is the location of the $j$th cluster. Since (8) is a conservation law $\sum_{j=1}^{\gamma} m_j = 1$.

## 3 Results and comparisons

We numerically analyze opinion formation dictated by equation (8) in the range $0.5 \leq W \leq 10$. We numerically integrate the master equation (8) for $W = 0.5, 0.52, 0.54, \ldots 10$. In Figure 1 we show the clusters' locations versus $W$. Opinion fragmentation increases with $W$. The first observed bifurcation occurs at $W_{\mathrm{A}} = 1.99 \pm 0.01$. For $W$ less than 1.99 the stubborn population achieves consensus. When $W$ reaches $W = 1.99$ the central opinion cluster is annihilated and two symmetric clusters nucleates in the proximity of the zero opinion. As $W$ increases the opinions of these symmetric parties slowly increases in a nonlinear way. The second bifurcation occurs at $W_{\mathrm{N}} = 2.75 \pm 0.01$. As $W$ achieves 2.75 we observe the nucleation of the central cluster, and the slopes of the symmetric clusters abruptly change. For $W$ larger than 2.75 the locations of these parties become linear with respect to $W$, at an approximate slope of 1. The bifurcation patterns is then periodically repeated. In Figure 2 we report the findings of [9,10] referring to acquiescent populations and based on (6) with $\mu = 1/2$. Interacting individuals either keep the initial opinions or agree on a compromising viewpoint. The bifurcation patterns in Figures 1 and 2 are qualitatively very different. First of all, persuasible societies segregate minorities while stubborn populations do not. Indeed, when $W = 1$, Figure 2 shows the nucleation of two extremist parties which are instead not present in Figure 1. This type of bifurcation is periodically repeated in Figure 2. In addition, the nucleation of the central cluster follow a different scheme for the two cases. In Figure 1 the nucleation of the central opinion does not correspond to the slope discontinuity in the symmetric clusters. Quantitatively, the nucleation of the central opinion occurs earlier for persuasible populations. Also the annihilation of the central opinion occurs earlier for persuasible societies. In Figure 2, the first bifurcation is approximately at $W = 1.87$.

For stubborn populations the mass of clusters decay hyperbolically. In Figures 3 and 4 we report the mass of the central cluster and of positive opinion cluster that nucleates at 1.99 versus the parameter $W$, respectively. During each nucleation-annihilation the mass of the central cluster increases approximately linearly and its maximum value hyperbolically decays over time. The nucleation of

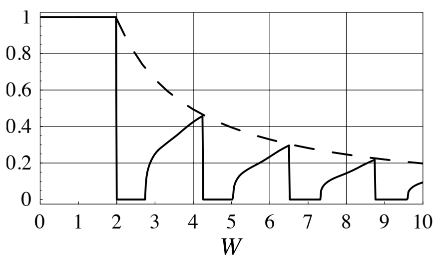

**Fig. 3.** Mass of the central cluster versus the initial opinion range $W$. The dashed line is the hyperbola $1.98/W$.

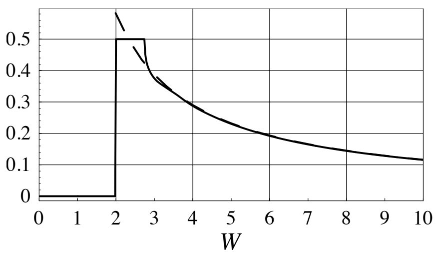

**Fig. 4.** Mass of the first bifurcated cluster versus the initial opinion range $W$. The dashed line is the hyperbola $1.16/W$.

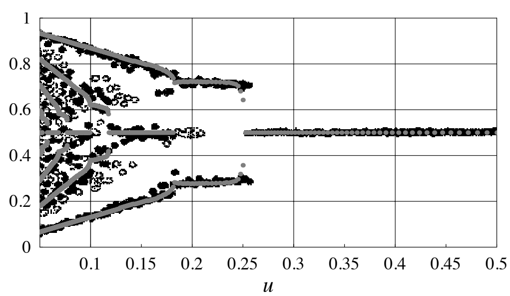

**Fig. 5.** Distribution of the normalized final opinions versus the normalized threshold $u$. Gray points refer to the mean field approximation. Black points and open circles refer to the Monte Carlo simulations in [1]. Each black point is a cluster larger than 10% of the total population.

the central cluster appears to be very sharp in contrast with the smooth behaviors found in [9,10] for persuasible societies.

In Figure 5, we compare the mean field model (8) with the Monte Carlo simulations of [1] using the discrete model in (1) and (2) with $\mu = 0.01$. The opinion profile is normalized in the range $[0,1]$. The mean field approximation seems in very good agreement with Monte Carlo findings and accurately describe the opinion formation for $u$ greater than 0.1. For smaller threshold it seems that a relatively higher number of agents is necessary for obtaining a sharp bifurcation pattern. We note that, in the underlying stochastic process described by (1) and (2), and corresponding Monte Carlo simulations, the agents are picked at random. Specifically, the agents are connected by a complete graph provided that their opinions are sufficiently close. The master equation (8) is descriptive of the limit of an infinite number of agents.

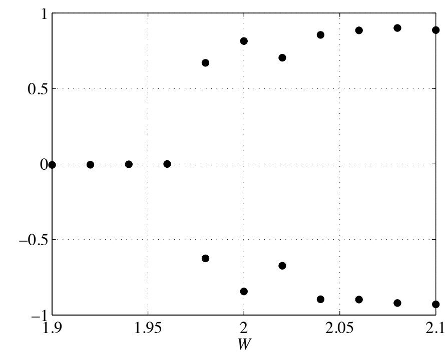

**Fig. 6.** Distribution of the normalized final opinions versus the initial opinion range $W$ for 10 000 agents and $\mu = 10^{-3}$. Each black point represents a population cluster.

## 4 A further insight into the loss of consensus

In order to better visualize the bifurcation pattern in the neighborhood of $W = 2$, we performed a numerical simulation using 10 000 agents and $\mu = 10^{-3}$. The initial opinions are uniformly assigned for each confidence level and a sufficiently large number of time steps is used for achieving full convergence. Numerical findings reported in Figure 6 show that all the opinions are clustered in either one central cluster or two symmetric opinion clusters. Comparing Figure 6 with Figure 5 we note that decreasing the parameter $\mu$ leads to a clear annihilation of the central cluster. We also note that apparently annihilation occurs very close to $W = 2$. This numerical evidence along with the results of our mean field model seem to indicate that for infinitely large populations consensus exists for any $W < 2$.

Even though we cannot provide a rigorous argument that in persuasible societies consensus is achieved for any $W < 2$, we present a heuristic argument that strengthens our numerical finding.

We define the opinion mass in the neighborhood of the opinion $\overline{x}$ at time $\tau$ by

$$M_0(\overline{x},\tau) = \int_{\overline{x}-1}^{\overline{x}+1} P(x,\tau)dx \qquad (11)$$

and the first opinion moment in the neighborhood of the opinion $\overline{x}$ at time $\tau$ by

$$M_1(\overline{x},\tau) = \int_{\overline{x}-1}^{\overline{x}+1} xP(x,\tau)dx. \qquad (12)$$

Therefore, using equation (8) and the definitions (11) and (12), the time rate of change of $M_0(\overline{x},\tau)$ can be written as

$$\begin{aligned}\frac{\partial M_0(\overline{x},\tau)}{\partial\tau} &= -P(\overline{x}+1,\tau)[M_1(\overline{x}+1,\tau)\\ &\quad -(\overline{x}+1)M_0(\overline{x}+1,\tau)] + P(\overline{x}-1,\tau)[M_1(\overline{x}-1,\tau)\\ &\quad -(\overline{x}-1)M_0(\overline{x}-1,\tau)]. \qquad (13)\end{aligned}$$

Similarly, the time rate of change of $M_1(\overline{x},\tau)$ can be expressed as

$$\frac{\partial M_1(\overline{x},\tau)}{\partial\tau} = -(\overline{x}+1)P(\overline{x}+1,\tau)[M_1(\overline{x}+1,\tau) - (\overline{x}+1)M_0(\overline{x}+1,\tau)] + (\overline{x}-1)P(\overline{x}-1,\tau)[M_1(\overline{x}-1,\tau) - (\overline{x}-1)M_0(\overline{x}-1,\tau)] + \int_{\overline{x}-1}^{\overline{x}+1} P(y,\tau)(M_1(y,\tau) - yM_0(y,\tau))dy. \quad (14)$$

Due to the spatial symmetry of the solution at any time $\tau$, we have

$$M_0(\overline{x},\tau) = M_0(-\overline{x},\tau), \quad \forall \overline{x} \in [-W, W] \quad (15)$$

$$M_1(0,\tau) = 0. \quad (16)$$

For $W < 2$ the opinions are located in the region $(-2, 2)$. That is

$$P(x,\tau) = 0, \quad \text{if} \quad |x| \quad \geq 2. \quad (17)$$

Using equation (17) and the symmetry condition (15), the following constraint holds

$$M_0(0,\tau) + 2M_0(2,\tau) = 1. \quad (18)$$

We note that $M_0(\overline{x},\tau)$ represents the mass of the opinion region $[-1,1]$, while $2M_0(2,\tau)$ signifies the total mass of the opinion region $[-W,-1] \cup [1,W]$. Evaluating equation (13) at $\overline{x} = 0$ and $\overline{x} = 1$ and accounting for the symmetry conditions (15) and (16) and equation (17) we obtain

$$\frac{\partial M_0(0,\tau)}{\partial\tau} = 2P(1,\tau)(M_0(1,\tau) - M_1(1,\tau)) \quad (19)$$

$$\frac{\partial M_0(1,\tau)}{\partial\tau} = 0. \quad (20)$$

Equation (20) implies that $M_0(1,\tau)$ is constant for all $\tau$, thus

$$M_0(1,\tau) = M_0(1,0) = 1/2. \quad (21)$$

Evaluating equation (14) at $\overline{x} = 1$ and accounting for equation (17) yields

$$\frac{\partial M_1(1,\tau)}{\partial\tau} = \int_0^2 P(y,\tau)(M_1(y,\tau) - yM_0(y,\tau))dy.$$

Using the definitions (11) and (12), the above quantity can be rewritten as

$$\frac{\partial M_1(1,\tau)}{\partial\tau} = \int_0^2 \int_{y-1}^{y+1} (z-y)P(y,\tau)P(z,\tau)dzdy. \quad (22)$$

The integration in equation (22) can be partitioned in three disjoint polygons as follows

$$\frac{\partial M_1(1,\tau)}{\partial\tau} = \int\int_{R_1} (z-y)P(y,\tau)P(z,\tau)dydz + \int\int_{R_2} (z-y)P(y,\tau)P(z,\tau) + \int\int_{R_3} (z-y)P(y,\tau)P(z,\tau)dydz$$

where the polygons $R_1$, $R_2$ and $R_3$ are defined by $R_1 = \{(y,z) \in (0,2)\times(0,2) : z \geq y-1, z \leq y+1\}$, $R_2 = \{(y,z) \in (0,1)\times(-1,0) : z \geq y-1\}$ and $R_3 = \{(y,z) \in (1,2)\times(2,3) : z \leq y+1\}$. Using equation (17) the integral over $R_3$ is equal to zero. Due to the symmetry conditions the integral over $R_1$ is equal to zero. In addition, since the probability function is positive and $y \geq z$ in $R_2$

$$\frac{\partial M_1(1,\tau)}{\partial\tau} \leq 0. \quad (23)$$

Since $M_1(1,0) = W/4$, the inequality above implies that for any instant $\tau$

$$\partial M_1(1,\tau) \leq W/4. \quad (24)$$

Using equations (21) and (24) in equation (19) we finally obtain

$$\frac{\partial M_0(0,\tau)}{\partial\tau} \geq P(1,\tau)(1 - W/2). \quad (25)$$

This equation implies that the mass of the central region cannot decrease over time, whereas, through equation (18) we can claim that the mass of the outer region cannot increase over time. Assuming that the final solution consists at most of two opinion clusters, we can infer that the clusters are located in the central region $[-1,1]$.

In order to show that the final solution features only the central opinion cluster, we evaluate (8) at the zero opinion

$$\frac{\partial P(0,\tau)}{\partial\tau} = P(0,\tau)(M_0(0,\tau) - 2P(1,\tau))$$

where we set $\frac{\partial P(0,\tau)}{\partial x} = 0$ and $P(1,\tau) = P(-1,\tau)$ due to the symmetry. Therefore, since $M_0(0,\tau)$ is not decreasing, the central opinion cluster will be present unless two opinion clusters arise at $x = \pm 1$. This last scenario is not feasible since according to (25) it will lead to an infinite value of the mass of the central region.

## 5 Conclusions

We studied opinion formation in a compromise model for stubborn individuals. Randomly selected individuals meet and if their opinions are within a given confidence threshold they accept to talk. The result of the information exchange is a mutual comprehension, that leads to a slight smoothing of the initial opinions. We derived a mean field approximation and we numerically integrated the resulting master equation. The solution of the master equation is in very good agreement with numerical findings based on Monte Carlo simulations. Stubborn societies are significantly different from persuasible populations. For stubborn societies there are only two types of bifurcations: splitting of the central party and re-nucleation of the central party. In persuasible populations, there are three, to include nucleation of the extremist minorities. Extremist minorities do not arise in stubborn societies. In addition, the population achieves consensus for smaller values of the confidence threshold. The significance of this work is that we have produced a new analytic model for the opinion formation problem in stubborn societies, which displays many of the features of the original model of [1], but now in an analytically amenable form.

## Appendix A: Numerical integration

Numerical integration of (7) is done by dividing the interval $[-W, W]$ into an even number $M$ of equal intervals of length $\epsilon$. We assume that the interval $[-1, 1]$ is consequently divided into $m$ even intervals, so that $\epsilon = 2/m$. Each interval is centered at $c_j = -\Delta + (j - 1/2)\epsilon$, $j = 1, \ldots M$. The probability function is approximated by a constant in each interval

$$P(x, \tau) = \sum_{j=1}^{M} p_j(\tau)\rho(x - c_j, \epsilon/2) \tag{26}$$

where $p_j(\tau)$ are unknown functions of time. Outside the interval $[-W, W]$, $P$ is zero. By using (26), the convolution in (8) at $c_i$ is approximated

$$(h * P)(c_i, \tau) = \sum_{j=1}^{M} a_{ij} p_j(\tau) \tag{27}$$

where

$$a_{ij} = \int_{c_i-1}^{c_i+1} y\rho(y - c_j)dy - c_i \int_{c_i-1}^{c_i+1} \rho(y - c_j)dy.$$

The term $a_{ij}$ in (27) may be analytically calculated

$$a_{ij} = \begin{cases} (j-i)\epsilon^2/2 & \text{if} \quad |i-j| < m/2 \\ \epsilon(\epsilon/8 + c_j/2) & \text{if} \quad i-j = m/2 \\ \epsilon(-\epsilon/8 + c_j/2) & \text{if} \quad i-j = -m/2 \\ 0 & \text{if} \quad |i-j| > m. \end{cases} \tag{28}$$

We approximate the space derivative in (8) using centered finite differences. In order to avoid high frequency oscillations in the numerical solution, we include a dissipative term in (27)

$$\frac{\partial P(x, \tau)}{\partial \tau} = -\frac{\partial}{\partial x}[P(x, \tau)(h*P)(x, \tau)] + d\frac{\partial^2 P(x, \tau)}{\partial x^2}. \tag{29}$$

Using (27) and imposing the probability function is zero outside the interval $[-W, W]$ from (29) we obtain the nonlinear system of coupled ordinary differential equations

$$\frac{dp_1(\tau)}{d\tau} = -\frac{1}{2\epsilon}\sum_{j=1}^{M} a_{2j} p_j(\tau) p_2(t) + \frac{d}{\epsilon^2}(p_2(\tau) - 2p_1(\tau))$$

$$\frac{dp_i(\tau)}{d\tau} = -\frac{1}{2\epsilon}\sum_{j=1}^{M}(a_{(i+1)j} p_{i+1}(\tau) - a_{(i-1)j} p_{i-1}(\tau))p_j(\tau) + \frac{d}{\epsilon^2}(p_{i-1}(\tau) - 2p_i(\tau) + p_{i+1}(\tau)) \tag{30}$$

$$\frac{dp_M(\tau)}{d\tau} = \frac{1}{2\epsilon}\sum_{j=1}^{M} a_{(M-1)j} p_j(\tau) p_{M-1}(\tau) + \frac{d}{\epsilon^2}(p_{M-1}(\tau) - 2p_M(\tau))$$

where $i = 2, \ldots M-1$ and all the initial conditions are $\epsilon/(2W)$. Equations (30) may be easily integrated using any standard method as Runge-Kutta or Adams-Moulton. In our numerical simulations we set $m = 100$ and $d = 5 \times 10^{-6}$, and we use the built-in function `NDSOLVE` of `Mathematica`©. Due to the high computational time, we analyze opinion formation until a maximum time $t_{\max} = 25 \times 10^3$. Our simulations are subject to the usual difficulty near the bifurcation points, due to loss of hyperbolicity at those points. So even though the exact positions of the bifurcations are difficult to locate precisely, the simulations give good confidence in the existence of the bifurcations due to the branches which we see away from the actual bifurcation points.

This work was sponsored by the National Science Foundation via grant IIS-0238092, and by the Office of Naval Research via grants N000140310444, N000140510780, and N000140510516. The authors would also like to thank Ms. Francesca Fiorilli for the help with the Monte Carlo simulations.

## References

1. M.F. Laguna, G. Abramson, D.H. Zanette, Complexity **9**, 31 (2004)
2. G. Weisbuch, G. Deffuant, F. Amblard, J.-P. Nadal, Complexity **7**, 55 (2002)
3. A. Baldassarri, U. Marini Bettolo Marconi, A. Puglisi, Europhys. Lett. **58**, 14 (2002)
4. E. Ben-Naim, P.L. Krapivsky, Phys. Rev. E **61**, 9909176 (2000)
5. R. Hegselmann, U. Krause, J. Artif. Soc. Soc. Simul. **5**, (2002)
6. S. Galam, S. Moscovici, Eur. J. Soc. Psych. **21**, 49 (1991)
7. R. Axelrod, *The Complexity of Cooperation* (Princeton University Press, 1997)
8. M. Granovetter, Am. J. Soc. **83**, 1420 (1978)
9. E. Ben-Naim, P.L. Krapivsky, S. Redner, Physica D **183**, 190 (2003)
10. E. Ben-Naim, P.L. Krapivsky, F. Vazquez, S. Redner, Physica A **330**, 99 (2003)
11. S. Fortunato, V. Latora, A. Pluchino, A. Rapisarda, Int. J. Mod. Phys. C **16**, 1535 (2005)
12. D. Neau, unpublished, 2000
13. G. Toscani, Comm. Math. Sci., forthcoming
14. L.C. Evans, *Partial Differential Equations* (American Mathematical Society, Providence 1998)