Controlling Invariant Density: An l^∞ Solution to the Inverse Frobenius-Perron Problem

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Abstract

In a previous paper, [1], we gave a new formalism to solve the Inverse Frobenius-Perron Problem (IFPP), to produce a dynamical system, uniformly nearby a given dynamical sytem, but with a drastically different and desirable invariant density. Our previous algorithm reduced the problem of producing the dynamical system G with desirable statistics $\beta(x)$, into a constrained optimization problem, which we solved in l^2. However, we pointed out that if this l^2 solution does not correspond to a useful solution, one could not conclude nonexistence. The l^∞ solution to the same optimization problem allows for a sharp existence-nonexistence theorem. In this paper, we present for the first time an l^∞ algorithm which produces solutions to our IFPP, and conclude our nonexistence theorem which is pertinant to this solution. Then we discuss applications in control of chaos, both by open-loop control strategies for maps, and we discuss future applications to feedback control of flows.

1. Introduction

A major difficulty of nonlinear control theory has involved globally representing the action of the dynamical system on its phase space. A complete, and now classical, global representation of a dynamical system is in terms of its symbolic dynamics. We have recently [1] demonstrated that a graph approximation of the symbolic grammar is a highly efficient way to completely encode course-grained control strategies. Closely related to the topological notions of symbolic dynamics, are the measure-theoretic notions of the transfer operator, including the Frobenius-Perron operator, which time-advances ensemble densities of initial conditions. The author has recently shown that since such global representations of these operators are easily course-grained by matrices (this idea dates to Ulam [4]), the difficult problems of developing a global control strategy can be reduced to much easier problems, 1) of linear algebra if targeting invariant-density (IFFP - "Inverse Frobenius-Perron Problem"), 2) or to symbolic dynamic [2] and combinatorial problems [3] of path searching, if targeting optimal trajectories. Essentially, the nonlinearity of "stretch-and-fold" can be accounted for by the action of a linear operator, approximated by a matrix, or equivalently by a digraph over the function space of densities. In this course-grained approximation, paths through graphs model epsilon-chain trajectories of the dynamical system.

Here, we extend control strategies based on transfer operator representations of the action of a chaotic dynamical system, on its invariant subspace. Suppose we are given a dynamical system $F: M \rightarrow M$, with an undesirable invariant density $\rho(x)$, but we prefer the long term statistics of some other measure $\tilde{\beta(x)}$. We wish to construct a sup-norm nearby dynamical system G such that this new dynamical system has the desirable statistics $\beta(x)$. In our previous work, 1) we approximated the Frobenius-Perron (F-P) operator of F, 2) we gave an algorithm which generates a new F-P operator, which when its exists, has the more desirable density as its steady state, and then 3) using what we call the "Inverse Ulam Problem," we constructed a transformation $\tilde{G}: M \rightarrow M$ which has the required F-P operator and hence the desirable statistics $\beta(x)$. Our algorithm was based on a constrained minimization problem, which we previously solved in l^2, by a modified least squares algorithm using Singular Value Decomposition (SVD), but we gave a theorem that stated that if this l^2 solution did not generate the desired statistics $\beta(x)$, then it was feasible that there still existed some as yet unfound solution. Then we pointed out that the l^∞ minimal solution would not allow such a loop-hole. Hence given the l^∞ solution of the same constrained optimization, not finding an appropriate F-P operator implies nonexistence of a transformation G uniformly nearby F, which has the desirable statistics. Therefore, in this paper we present for the first time a new algorithm, based on repeated linear programming, to solve the required constrained optimization l^∞.

We consider the implication of being able to globablly stabilize a desirable invariant state, or drammatically alter long term operating properties of a chaotically oscillating circuit to be important, opening a rich range of potential new applications. As a preview of such potential, we consider the doublewelled Duffing system as an example of our techniques applied to a flow system, and other flow systems such as say, Chua's circuit are equally readily altered.

2. IFPP: Targeting Invariant Measure

Remarkably, while a chaotic dynamical system

$$F: M \rightarrow M, M \in R^n \tag{1}$$

is nonlinear and typically has the critical ingredient of chaos, "stretch-and-fold," analogous to the Smale horseshoe [5], the one-step action of the map in the space of (ensembles of initial conditions) densities is that of a linear transfer operator [7, 8, 9]. The Frobenius-Perron operator generates an associated linear dynamical system on the space of densities.

$$P_F: L^1(M) \rightarrow L^1(M) \tag{2}$$

defined by [10, 7, 8],

$$(P_F)\rho(x) = \sum_{\{y: F(y)=x\}} \frac{\rho(y)}{|F'(y)|} = \int_M \delta(y - F(x))\rho(y)dy, \tag{3}$$

where the sum is taken over all pre-images, y, when the map has a multiply branched inverse. This infinite dimensional operator is typically not realizable in closed form, except for special cases [8]. However, the so-called Ulam's method, conjectured in 1960 by S. Ulam [4], and proven for specific cases (see for example [11, 12, 13, 14]), provides a robust technique to project this operator to a finite dimensional linear subspace of $L^1(M)$ generated by the set characteristic functions supported over the partitioning grid [11]. The idea is that refining the grid yields weak approximants of invariant density. Roughly speaking, the infinitesimal transfer operator, [15], $\mathcal{L}(y, x) = \delta(y - F(x))$, integrated over a grid square B_i, which is small enough so that $F'(x)$ is approximately constant, is approximated by a constant matrix entry $A_{i,j}$. Under special assumptions on $F(x)$, statements concerning quality of the approximation can be made rigorous. Recently, many groups have been using Ulam's method to describe global statistics of a dynamical system, [12, 13, 14, 16, 17] such as invariant measure, Lyapunov exponent, dimension, etc. *From our point of view, this Ulam approximation of the F-P operator, and other similiar transfer operators, are an ideal starting point to realize a wide range of global control objectives.*

In [1], we considered the following IFFP. Given a dynamical system Eq. (1), which has an undesirable "typical" invariant density $\rho(x) \in L^1(M)$, but we prefer a density $\beta(x)$ instead. (While an arbitrary DS is not know to have an SBR invariant measure, this is not a problem since we will be controlling it away.) Stated as an open-loop control problem, we asked in [1], can a dynamical system G, uniformly nearby F, be found such that G has β as its invariant density?

Our approach was, based on Ulam's method, to approximately discretize the one step transfer action of the dynamical system. First we choose a uniform ϵ-grid, covering the attractor, which is made-up of an $N \times N \times ... \times N$ grid of squares $\{Q_i\}$. Choosing ϵ allows us to contol the sup-norm deviation between G and F over M. Now, using a long test orbit, of a "randomly" selected initial condition, whose orbit we presume to be dense in the attractor, we can build a transition matrix A of *observed* transitions of orbits through the grid,

$$A_{i,j} = \frac{\#\{x_k \text{ such that } x_k \in Q_j \text{ and } f(x_k) \in Q_i\}}{\#\{x_k \in Q_j\}} \tag{4}$$

This formula, for long test orbits, and small squares, gives approximately a finite-dimensional (matrix) approximation of the true infinite dimensional F-P operator. Said another way, the matrix can be considered as a finite dimensional projection of the F-P operator, into the finite dimensional linear subspace of $L^1(M)$, generated by characteristic functions $\chi_{A_i}(x)$ supported over each square Q_i. Typically the grid is not completely occupied (consider the box-counting dimension).

Stated in terms of the transition matrix A, which has a steady state dominant eigenvector v (which approximated invariant density $\rho(x)$), our IFPP requires us to construct a new matrix $\tilde{A}$ such that the new steady state dominant eigenvector $\tilde{v}$ approximates the desired density $\beta(x)$. We formally define the following optimization problem to find a desired matrix $\tilde{A} = A + \delta A$,

Control Rules:

1. $(A + \delta A) \cdot (\mathbf{v} + \delta \mathbf{v}) = \mathbf{v} + \delta \mathbf{v}$, forces the desirable steady state.
2. $(A + \delta A)$ is stochastic. Therefore,
   (a) $\sum_{i=1}^{q} (A + \delta A)_{i,j} = 1$, for all j,
   (b) and $0 \leq (A + \delta A)_{i,j} \leq 1$ for all (i, j).
3. Preserve grammar in the sense that we require $\delta A_{i,j} = 0$ for each (i, j) that $A_{i,j} = 0$, which controls maximal variations $\|F(x) - G(x)\|_{\sup(M)} \leq \epsilon$.
4. $\|\delta A_{i,j}\|$ is as small as possible.

Rules 2-4 serve as constraints on Rule 1. Constraints 2(b) is a *posterior* condition for evaluating success of a given solution. Notice the original steady state solution was $A \cdot \mathbf{v} = \mathbf{v}$, and therefore subtracting this from both sides of Rule 1 gives

$$\delta A \cdot (\mathbf{v} + \delta \mathbf{v}) = (I - A) \cdot \delta \mathbf{v}. \tag{5}$$

Considering which are the variables and which are the known values reveals that Eq. (5) is written "backwards." Here, the matrix A is known (found by Ulam's method) and hence $\mathbf{v}$ is easily calculated. Likewise, $\mathbf{v} + \delta \mathbf{v}$ is an approximation of our desired target density $\beta(x)$ which we know. What we don't know is the required variations δA of the matrix. As is more usual, this equation can easily be rewritten to emphasize the unknown values (the nonzero entries of the matrix δA) as a vector $\delta \mathbf{a}$ (indexed arbitrarily),

$$\delta A \cdot \mathbf{x} = \mathbf{y}, \tag{6}$$

$$\text{where, } \mathbf{x} = \mathbf{v} + \delta \mathbf{v}, \text{ and} \mathbf{y} = (I - A) \cdot \delta \mathbf{v}. \tag{7}$$

In [1] we gave a specific example to help clarify the role of the q-variables, due to the T-nonzero entries of δA, which is $m \times m$, and therefore $T \leq m^2$, but typically much less dues to typically small topological entropy. Therefore, we can generally rewrite Eqs. (6), together with the linear constraint that $A + \delta A$ be stochastic, Rule 2(a) has the following solution hyperplane,

$$\overset{2m \times T}{\begin{pmatrix} B \\ - \\ C \end{pmatrix}} \cdot \overset{T \times 1}{\boldsymbol{\delta a}} = \overset{2m \times 1}{\begin{pmatrix} \mathbf{y} \\ - \\ \mathbf{0} \end{pmatrix}}, \tag{8}$$

or simply write the generic linear equation,

$$D \cdot \boldsymbol{\delta a} = \boldsymbol{z}, \text{ where, } D = \begin{pmatrix} B \\ - \\ C \end{pmatrix} \text{ and } \boldsymbol{z} = \begin{pmatrix} \mathbf{y} \\ - \\ \mathbf{0} \end{pmatrix} \tag{9}$$

We call this a solution hyperplane since we are typically underconstrained, $2m < T$, as D is not of full rank, leaving infinitely many solutions. We are now in a position to carefully state the new result in this paper.

This underdetermination of our inverse problem is in fact desirable, as it allows freedom in choosing a solution. We choose to find the solution on the hyperplane $D \cdot \boldsymbol{\delta a} = \boldsymbol{z}$ subject to linear constraints that A is stochastic (Rule 2) which is closest to A, so $\boldsymbol{\delta a}$ is minimal. Previously, we used the Penrose-pseudo inverse method to determine the least squares solution of Eq. (9), $\boldsymbol{\delta a} = D^+ \mathbf{z}$, where the Singular Value Decomposition $D = U \cdot \Omega \cdot V^t$ [18], gives $D^+ = V \cdot \Omega^{-1} \cdot U^t$. While this l^2 solution is usefully constructive, and straight-forward to calculate via well known algorithms, it does not allow non-existence conclusions if this l^2 solution of the constrained optimization problem, Rules 1-4, is not a good solution. In [1], we called this the "round-peg square-hole" problem. Basically, constraints of Rule 2(b) define a hyper-cube, but least-squares defines round balls, so while the least squares solution could be outside of the "stochastic box" defined by Rule 2(b), there could still exist a solution lurking in a corner of the hyper-cube, which least-squares cannot rule-out. The l^∞ solution does not suffer this problem. Thus, we are motivated to present the l^∞ algorithm for the first time in this paper.

3. Control by "Box" Norm

In the last section, we reviewed that control problem Rules 1-4, defines a linearly constrainted optimization problem, which is easily solved, when a solution exists, by least squares, but we remarked that nonexistence of this solution cannot rule out the existence of some solution to optimization problem Rules 1-4, and therefore neither to the IFPP. Since the l^∞ solution (the "box" norm") does not suffer the same restriction, since the linear constraints define a hyper-cube. We now present for the first time, this l^∞ solution, from which we can state the sharp existence theorem at the end of this section.

We have designed a simple algorithm, based on repeated linear programming [19], to construction the l^∞ minimal solution of the control problem 1-4. The objective function is chosen to be the infinity-norm,

$$n(\boldsymbol{\delta a}) = \max_i |\delta a_i|, \ \boldsymbol{\delta a} \in H \tag{10}$$

where H is the set of all $\boldsymbol{\delta a}$ satisfying the two sets of constraints, $D \cdot \boldsymbol{\delta a} = \boldsymbol{z}$ and Rule 2(b), which we rewrite,

$$D \cdot \boldsymbol{\delta a} = \boldsymbol{z}, \tag{11}$$

$$-a_k \leq \delta a_k \leq 1 - a_k, \text{ for } 1 \leq k \leq T. \tag{12}$$

The objective function Eq. (10) is not linear, and cannot be minimized directly by linear programming, but can easily be adapted to apply linear programming. Consider the objective function which selects the i^{th} component of δa_i,

$$n_i(\boldsymbol{\delta a}) = \delta a_i, \ \boldsymbol{\delta a} \in H \cap K_i, \tag{13}$$

when the i^{th} component is maximal, which is the additional constraint,

$$K_i = \{\boldsymbol{\delta a} : \delta a_i - \delta a_j \geq 0\}. \tag{14}$$

In this setting, each n_i is a linear objective function, subject to linear constraints $H \cap K_i$, and therefore we can easily find the minimizing $\boldsymbol{\delta a}$, by linear programming [19]. Linear programming is a well matured topic, and highly robust routines are readily available; we have used both the routines built into *Mathematica*, and the *Numerical Recipes* subroutines [19]. First, we must shift the problem,

$$\tilde{\delta a}_k = \delta a_k + a_k, \tag{15}$$

to a unit hyperbox which allows us to avoid the (nonlinear) absolute value sign. Therefore, we replace the objective function (13), and constraints (11), (12) and (14) with,

$$\tilde{n_i}(\boldsymbol{\delta a}) = \delta a_i, \text{ subject to } \boldsymbol{\delta a} \in \tilde{H} \cap \tilde{K_i}, \tag{16}$$

$$\tilde{H} = \{\tilde{\boldsymbol{\delta a}} : D \cdot \tilde{\boldsymbol{\delta a}} = \tilde{\boldsymbol{z}}, \text{ and } 0 \leq \tilde{\delta a}_k \leq 1, \text{ for } 1 \leq k \leq T\} \tag{17}$$

$$\tilde{K_i} = \{\tilde{\boldsymbol{\delta a}} : \tilde{\delta a}_i - \tilde{\delta a}_j \geq 0\}. \tag{18}$$

Note that substituting the shift (15) into constraint (11) gives the shifted data vector $\tilde{\boldsymbol{z}} = \boldsymbol{z} + D \cdot \boldsymbol{a}$.

Given the T minima of $\tilde{n_i}$, the overall minimum yields the minimum of the infinity-norm Eq. (10), once this minimum has been unshifted by Eq. (15),

$$\min_{\boldsymbol{\delta a} \in H} n(\boldsymbol{\delta a}) = \left[\min_{1 \leq i \leq T} \left(\max_{\boldsymbol{\delta a} \in \tilde{H} \cap \tilde{K_i}} \tilde{n_i}(\boldsymbol{\delta a}) \right) \right] - \boldsymbol{a}, \tag{19}$$

and this can easily be calculated by iterating the linear programming routines T times and then selecting the minimum, largest $\tilde{n_i}$.

Now given this l^∞ solution, we can recall our [1] sharp nonexistence theorem, which is now useful.

Theorem: Given the stochastic matrix A, with stationary distribution $\mathbf{v}$, then if the l^∞ minimal solution of $D \cdot \boldsymbol{\delta a} = \boldsymbol{z}$ found according to the linear programming algorithm Eqs. (16)-(19), yields a $A + \delta A$ which *is not stochastic, then no such* $A + \delta A$ *exists*

Given the above newly presented optimization algorithm, and recalling our above theorem, we have a true constructive procedure to either find a good solution, or to conclude that none exists.

4. Back to Dynamical Systems

After the above forary into linear algebra, and then optimization, we now recall our original problem. Find a transformation $G : M \to M$ whose statistics are desirable. Given a successful solution $(A + \delta A)$ of Rules 1-4, we must construct a piece-wise transformation, on the grid squares $\{Q_i\}$, whose $F - P$ operator is *exactly* the new matrix $(A + \delta A)$. We call this the "Inverse Ulam Problem." We a detailed description of how to construct a piece-wise transformation in [1], the details of which we do not have space to reproduce here, but the main idea of the construction is as follows. A stochastic transition matrix is equivalent to a Markov directed graph. Note also that the definition of an F-P operator Eq. (2) has reciprocals of derivatives (or determinant of the Jacobian) in the formula. Therefore, the idea is to produce a piece-wise linear (affine) function whose local determinant Jacobians are reciprocals of entries of $(A + \delta A)$. Said simply, if $(A + \delta A)_{i,j} = 0.1$ for example, then its is necessary that exactly 10% of box Q_j be mapped into Q_i. Furtheremore, for reasons of requiring a Markov transformation [6], it is necessary to require that horizontal strips stretching all the way across Q_j be mapped exactly onto some vertical strip which stretches all the way across Q_i.

Given a stochastic matrix A, we construct a piecewise affine Markov map. Basically, we map $100 \cdot A_{j,i}$% of the grid-cell Q_i onto $100 \cdot \frac{A_{j,i}}{\sum_k A_{j,k}}$% of the cell Q_j. This is done by the affine transformation,

$$\boldsymbol{f}\,_n^{i,j}(x,y) = \begin{pmatrix} \frac{\Delta x'_{j,i}}{\Delta x_{j,i}} & 0 \\ 0 & \frac{\Delta y'_{j,i}}{\Delta y_{j,i}} \end{pmatrix} \cdot \begin{pmatrix} x - x_{j,i} \\ y - y_{j,i} \end{pmatrix} + \begin{pmatrix} x'_{j,i} \\ y'_{j,i} \end{pmatrix}$$

which simply scales the $\Delta x_{j,i} \times \Delta y_{j,i}$ rectangle $R_{j,i}$, whose lower left corner is $(x_{j,i}, y_{j,i})$, linearly onto the $\Delta x'_{j,i} \times \Delta y'_{j,i}$ rectangle $R'_{j,i}$, whose lower left corner is $(x'_{j,i}, y'_{j,i})$. Our notation convention is that $R_{j,i} \subset Q_i$ is the rectangle in Q_i which maps *into* Q_j, but *onto* $R'_{j,i} \subset Q_j$, where $R'_{j,i}$ is the part of Q_j that came from Q_i. The situation is even simpler for 1-D transformations. These transformations can be considered to be highly generalized Baker's transformations, which are "open-loop" solutions to the control problem of selecting invariant measure.

5. Apriori Considerations, Examples

A well known theorem states [7, 1] that when a sequence of transformations $\{F_n\}$ has a uniform limit F, and the corresponding sequence of invariant pdf's $\{\rho_n(x)\}$ has a weak limit $\rho(x)$, then that invariant pdf must be F invariant. This places strong restrictions on the type of solution to our control problems we can expect, but it is nonetheless not a continuity-type theorem, even though it may seem like one at first reading. The theorem implies that if our target measure is already an F-invariant measure (but not necessarily the "typical", i.e., "natural" or SBR-invariant measure), then ϵ may be chosen arbitrarily small, and we can still find an ϵ-nearby to F transformation G (in the sup-norm) such that G has the new desired measure. Such target invariant measures must be supported over invariant sets, such as Cantor sets, and the delta measures supported over UPO's. In other words, we can make a UPO globally attracting for an arbitrarily close transformation G. Likewise, invariant Cantor sets which avoid some undesirable set allow for "anti-control" [20]; say in some practical application, we do not care were in phase space a trajectory goes, as long as it *does not go* to a region representing "bad" momenta or positions. On the other hand, if we attempt to target a measure which is not F-invariant, then the converse of the theorem states that there exists some critical minimal $\epsilon_{cr} > 0$ such that any attempt to solve the IFPP for $\|F - G\|_{\sup(M)} < \epsilon < \epsilon_{cr}$ must yield no solution to Rules 1-4, which is in fact what we do find in practice.

We now give two examples. Consider a 1-D example in which we are given the logistic map, $F(x) = 4x(1-x)$, and for this well known example, and exactly this parameter, we have the rare situation where we can exactly write down the invariant density, which is, $\rho(x) = 1/\pi\sqrt{x(1-x)}$ [8]. However, suppose we do not like this invariant density. We wish to choose an invariant density with zero support over the interval $[0.44, 0.58]$ (and therefore supported over the Cantor set which also does not include all pre-images of this interval). We show a piecewise linear transformation, and its invariant density in Fig.1.

As a second example, we show in Fig. 2 the invariant density of a piece-wise affine transformation which is nearby the Poincare' map of the Duffing oscillator, and whose invariant measure is atomic and supported over a fixed point of the Duffing map.

While the control strategy described here and in [1] is "open-loop" as stated in this paper, and in [1], in the context of maps, this type of control applies to flows, by considering the above calculated "nearby" maps on surface of section to be a set of globally calculated targeted next responses. Desirable statistics of maps or flows may be global stabilization of UPOs, anti-control of "bad sets" or targeting some arbitrary target measure. When F is a Poincare' map of a flow, then small variations $\|F-G\|_{\sup(M)}$ should be realizable by likewise small parameteric variations, of enough adjustable parameters (in Fig. 2 of the Duffing oscillator for example, we find that *two* variations, $(\delta a, \delta b)$ usually span the tangent space). Preliminary results indicate that a closed-loop control program globally mapped by the pre-calculated "targets-function" G is likely to be a practical new avenue to achieve a wide range of new control objectives. See Fig. 3 which is a characature of the flow of a Duffing oscillator, in which the precalulated variations between 2π-stroposcopic mappings might be forced according to control map G, and variations are guaranteed to be small by our construction.

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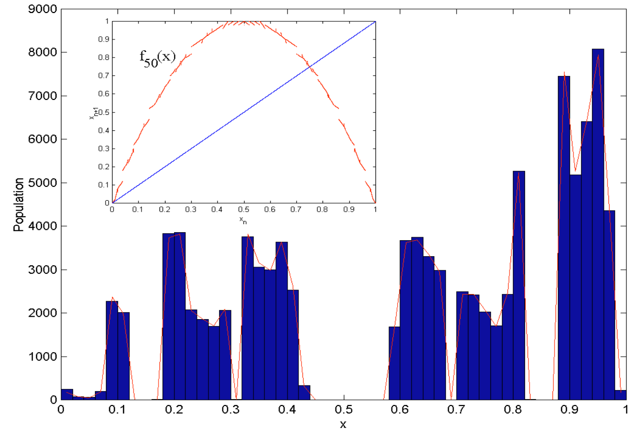

Figure 1: A piece-wise linear transformation, and its invariant density, constructed to avoid the interval $[0.44, 0.58]$, using a grid of 50-points. Note that this transformation looks strikingly like the logistic map (by construction) but close inspection reveals the differences.

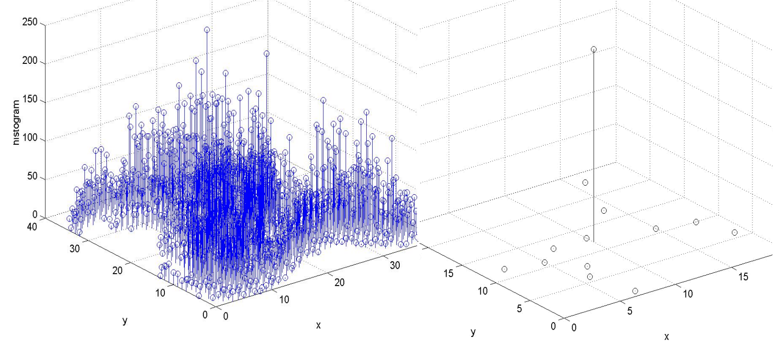

Figure 2: Left: Invariant density of the Duffing oscillator: $x'' + ax' + x^3 - x = b\sin(t)$. Right: Nearby the Duffing oscillator, (choose a 40×40 grid, on Poincare' surface of section, there exists a piece-wise affine map whose invariant measure is atomic, and supported over a fixed point. Such transformations can be found by the algorithm described here in

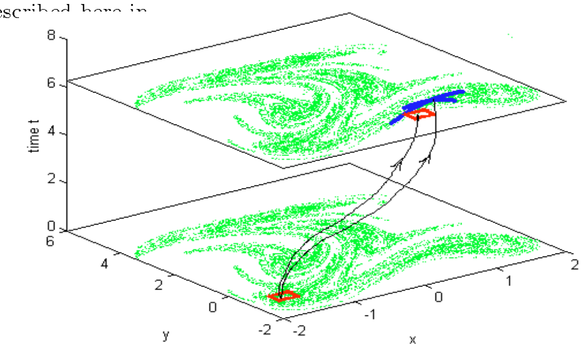

Figure 3: Characature of small variations in Duffing oscillator, according to uniformly nearby control map G (see Fig. 2), which can be achieved by small parameter variations. In this case, $x'' + ax' + x^3 - x = b\sin(t)$, with $(|\delta a|, |\delta b\|) < (0.02, 3)$, produce the transverse set of attainable next responses on the upper attractor, which allows for next responses not in the uncontrolled box Q_i (the box shown).