# How fragile is your network? More than you think.

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Graphs are pervasive in our everyday lives, with relevance to biology, the internet, and infrastructure, as well as numerous other applications. It is thus necessary to have an understanding as to how quickly a graph disintegrates, whether by random failure or by targeted attack. While much of the interest in this subject has been focused on targeted removal of nodes, there has been some recent interest in targeted edge removal. Here, we focus on how robust a graph is against edge removal. We define a measure of network fragility that relates the fraction of edges removed to the largest connected component. We construct a class of graphs that is robust to edge removal. Furthermore, it is demonstrated that graphs generally disintegrate faster than would be anticipated by greedy targeted attack. Finally it is shown that our fragility measure as demonstrated real and natural networks.

### I. INTRODUCTION

Complex networks can be found in many areas of our lives, from the brain [1–4], to our infrastructure [5–8] to our social interactions [9, 10] among others. Given how central they are, a question which may arise is how fragile/robust are these networks to lost edges or nodes?

As an example, when hurricane Irene ravaged the eastern coastline in 2011, one region in northern New York was effectively cut off from aid, due to the destruction of NY 73 [11]. In Germany in 2006 a single high voltage power line was shut off to allow a cruise ship to pass, triggering a power outage for millions of people [12].

While the above situations were quite different, it is clear that an understanding of what makes a complex network robust to failure is necessary to avoid potentially catastrophic situations. In the present work we will focus on the fragility of a graph to edge removal.

It is not always evident that a network will be fragile to edge removal. In the cases above the fragility could be attributed to the sparsity of the network, that is the vast majority of possible edges available are not realized within the graph. However, even relatively dense networks may be quite fragile, a simple example of this is shown in Fig. 1. The removal of a single edge is capable of splitting the graph into two equal size components. The single

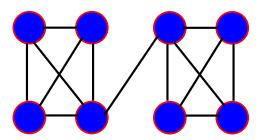


FIG. 1: An example of a graph that is very fragile to edge removal, despite being fairly dense.

edge connecting two highly connected components can be thought of as a bottleneck, and the graph thus has a small Cheeger constant [13, 14].

The example shown in Fig. 1 gives the overall sense of what fragile means, but the term fragility remains fuzzy. Intuitively, if only a "few" edges must be removed to break the network into "small" pieces, then we will call a network fragile in terms of edge removal. In order to make clear what is meant by "few" and "small", the central question we will be asking in this work is the following: what fraction of edges must be removed for the resulting graph to have some fraction of nodes remain in the largest connected component? It is through this question we will make clear what we mean by the term fragility.

The remainder of this work is laid out as follows. We will begin by offering a definition of *fragility*, relative to the fraction of edges which have been removed from the original graph (network). Next we define *robust* graphs, and follow with a proof that a certain class of graphs is robust. Finally we will investigate how fragile some real networks are under the newly defined measure.

#### II. NETWORK FRAGILITY

Throughout this paper we will use the terms graph and network interchangeably. We begin by offering a definition of a graph.

**Definition II.1.** Graph: A graph G = (V, E) is a set containing n vertices  $V = \{v_1, v_2, ..., v_n\}$  and a set of edges  $E \subseteq (V \times V)$ .

Graphs can be broken into directed and undirected graphs, based upon whether the edges are oriented.

**Definition II.2.** Undirected graph: An undirected graph is a graph in which if  $(v_i, v_j) \in E$ , then  $(v_j, v_i) \in E$ .

We will be exclusively exploring undirected graphs and  $(v_i, v_j)$  along with  $(v_j, v_i)$  will be counted as a single edge. As we are interested in how a graph changes with removed edges, we define a *perturbed graph* below.

**Definition II.3.** Perturbed graph: A perturbed graph G'(r) = (V, E') of G is a graph with r subtracted edges such that,

$$E \cap E' = E',\tag{1}$$

and

$$\operatorname{card}(E/E') = r, (2)$$

where  $\operatorname{card}(\cdot)$  is the cardinality of a set. We call a graph *connected* if for every disjoint partition of nodes there is at least one edge which joins nodes from different partitions. Not all graphs are connected however and thus we will define the *largest connected component* of a graph.

**Definition II.4.** Largest connected component: Begin with a partition of V into two disjoint sets  $W, X \subset V$ , with

$$W \cup X = V, \tag{3}$$

$$W \cap X = \emptyset. \tag{4}$$

Let

$$E' = (W \times W) \cap E, \tag{5}$$

and G' = (W, E'). G' is called a component of G. Furthermore, noting that a simple relabeling of the nodes of G remains the same graph G, let  $W = \{v_1, ..., v_k\}$  and  $X = \{v_{k+1}, ..., v_n\}$ . Finally, let,

$$C = \{x, y | (x, y) \in E'\}. \tag{6}$$

The largest connected component LCC(G) of G is the component G' with largest card(W) over all possible node relabelings, with,

$$v_i \in C \ (\forall i \in \{1, ..., k\}). \tag{7}$$

In other words, LCC(G) is the largest possible component in which all nodes of the component have at least one edge.

Certain classes of graphs are amenable to analysis because of their predictable structure. One such class of graphs are complete graphs which will now be defined.

**Definition II.5.** Complete Graph: A graph G is called *complete* if the edgeset of the graph is given by  $E = (V \times V)/\{(v_i, v_i)\}$ , and the complete graph on n nodes will be denoted by  $K_n$ . The number of edges of  $K_n$  will be written as  $\operatorname{card}(E(K_n)) = \frac{n(n-1)}{2}$ , where  $E(\cdot)$  is the edge set of a graph.

The central question that we will examine throughout this paper is how quickly a graph falls apart after targeted edge removals. By fall apart, we mean that a component disintegrates to smaller components. Suppose one wants to know what the minimal number of edges required to be removed from  $K_n$  in order to have a connected component of no larger than size c. As it turns out, the most effective strategy is to split the graph into as many components of size c and a single component of size b < c to remove the minimal number of edges.

We now require the following result as a preliminary for what follows.

Theorem II.1. Splitting squares

Let  $c > d_0 \ge d_1 \ge ... \ge d_l \ge d_{l+1} = 0$ ,  $c, d_i \in \mathbb{N}$   $(\forall i), a_0 = c - d_0, a_i = d_{i-1} - d_i$   $(\forall i > 0)$  and  $\sum_{i=0}^{l+1} a_i = c$ . Then

$$c^{2} \ge (c - d_{0})^{2} + \sum_{i=1}^{l-1} (d_{i-1} - d_{i})^{2} + d_{l}^{2}.$$
(8)

In other words splitting c into any number of integers will always result in a sum of squares which is less than or equal to  $c^2$ .

*Proof.* It is clear that:

$$(c+d_0)(c-d_0) \ge (c-d_0)(c-d_0), \tag{9}$$

since  $c \geq d_i \geq 0$ . Then

$$c^2 - d_0^2 \ge (c - d_0)^2 \implies c^2 \ge (c - d_0)^2 + d_0^2.$$
 (10)

Now suppose we split c with another term, then it is clear that:

$$(c-d_0)^2 + (d_0+d_1)(d_0-d_1) \ge (c-d_0)^2 + (d_0-d_1)(d_0-d_1) \implies c^2 \ge (c-d_0)^2 + d_0^2 \ge (c-d_0)^2 + (d_0-d_1)^2 + d_1^2$$
(11)

Finally suppose we split c, (l+1) times, we have:

$$c^{2} \ge (c - d_{0})^{2} + (d_{0} - d_{1})^{2} + \dots + (d_{l-1} + d_{l})(d_{l-1} - d_{l}) \ge (c - d_{0})^{2} + (d_{0} - d_{1})^{2} + \dots + (d_{l-1} - d_{l})(d_{l-1} - d_{l}), (12)$$

which can be rewritten as Eq. 8.

Exploiting Thm. II.1, the method outlined above can be shown to be the most efficient edge removal strategy for complete graphs.

Theorem II.2. Efficient destruction of Complete graphs via edge removal

Let the graph  $K_n$  be a complete graph, and let  $LCC(K'(r)_n)$  be the largest component of the perturbed graph  $K'(r)_n$ . Set  $r_{complete}^*(c) = \min_r(card(LCC(K'(r)_n)) = c)$  with c < n. Then

$$r_{complete}^{*}(c) = card(E(K_n)) - \left(\left\lfloor \frac{n}{c} \right\rfloor card(E(K_c)) + card(E(K_b))\right), \tag{13}$$

where  $b = (n \mod c)$  and  $|\cdot|$  is the floor function.

*Proof.* Let  $m = \operatorname{card}(E(K_n))$ ,  $b = (n \mod c)$  with c the number of nodes in the largest connected component, and the number of edges remaining after r removals be q. Clearly

$$r = m - q. (14)$$

Note that since  $E(K_n) = \frac{n^2 - n}{2}$  that  $E(K_n)$  grows as  $n^2$ . Suppose that b = 0, in other words suppose that c evenly divides n. Then  $\frac{n}{c} \operatorname{card}(E(K_c))$  is the largest possible number of edges remaining in the graph, and

$$r^*(c) = m - \frac{n}{c} \operatorname{card}(E(K_c)). \tag{15}$$

Eq. 15 follows from Thm. II.1 and from the fact that  $\operatorname{card}(E(K_n))$  grows as  $n^2$ . Now suppose that b > 0 and note that b < c and also note that  $n = \left\lfloor \frac{n}{c} \right\rfloor c + b$ . Now both b and c can be split and combined in any manner to form a sum of squares which may be written:

$$e = \sum_{i=0}^{l} a_i^2, \tag{16}$$

with  $\sum_{i=0}^{l} a_i = \left( \left\lfloor \frac{n}{c} \right\rfloor c + b \right)$  and  $0 \leq a_i \leq c \ (\forall i)$  since c is the largest allowable connected component. We may now

order the terms such that  $\sum\limits_{i=0}^{l-k}a_i=\left\lfloor\frac{n}{c}\right\rfloor c$  and  $\sum\limits_{l-k+1}^{l}a_i=b,$  with  $a_i\leq b$   $(\forall i\geq [l-k+1])$ . Then

$$e \le b^2 + \left\lfloor \frac{n}{c} \right\rfloor c^2, \tag{17}$$

follows from Thm. II.1. This implies that:

$$r^*(c) = m - \left( \left\lfloor \frac{n}{c} \right\rfloor \operatorname{card}(E(K_c)) + \operatorname{card}(E(K_b)) \right), \tag{18}$$

for the complete graph.

We assert that the complete graph is the least fragile in terms of edge removals. Thus in theory, in situations where one is concerned with the failure of a graph due to edge removals, one would design the graph of the system to be a complete one. However this is impractical in real world situations, often due to financial constraints as well as the enormity of the graphs at hand that tend to favor sparsity. For instance in the US there are approximately 20,000 cities, meaning that to connect every city directly to every other, it would be necessary to build  $\approx 400,000$  roads. Building and maintaining these roads would be prohibitively expensive. Instead, it is desirable to find graphs which are less dense, but which have a similar level of stability against edge removals. One such graph, which we will call the complete equitable bipartite graph, is presented in Fig. 2.

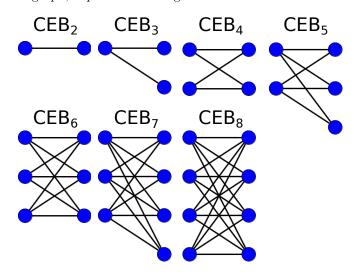


FIG. 2: Examples of CEB graphs. For even n we can see that there are  $\left(\frac{n}{2}\right)^2$  edges, while for odd n the number of edges grows as  $\left(\frac{n+1}{2}\right)^2 - \frac{n+1}{2}$ .

# **Definition II.6.** Complete Equitable Bipartite (CEB) Graph:

We call a graph G with number of nodes n a complete equitable bipartite (CEB) graph if the graph is partitioned into two disjoint sets of nodes  $W \cup X = V, W \cap X = \emptyset$  such that the cardinality of W and X differs by at most 1 (that is  $|\operatorname{card}(W) - \operatorname{card}(X)| \le 1$ , where  $|\cdot|$  is the absolute value and if the graph has edgeset  $E = (W \times X) \cup (X \times W)$ .

The CEB graph on n nodes will be denoted by CEB<sub>n</sub>. Note that,

$$\operatorname{card}(E(\operatorname{CEB}_n)) = \begin{cases} \left(\frac{n}{2}\right)^2, & \text{for } n \text{ even,} \\ \left(\frac{n+1}{2}\right)^2 - \frac{n+1}{2}, & \text{for } n \text{ odd,} \end{cases}$$
 (19)

as can be seen in Fig. 2. Since  $\operatorname{card}(E(\operatorname{CEB}_n)) \propto n^2$ , we can follow the logic of Thm. II.2 and find that:

$$r_{\text{CEB}}^*(c) = \operatorname{card}(E(\text{CEB}_n)) - \left(\left\lfloor \frac{n}{c} \right\rfloor \operatorname{card}(E(\text{CEB}_c)) + \operatorname{card}(E(\text{CEB}_b))\right)$$
 (20)

In order to make a comparison of graphs, we define the fragility of a graph to edge removals.

# **Definition II.7.** Fragility:

Let G be a graph,  $\delta = \frac{c}{\operatorname{card}(G)} < 1$  be the fractional component size and  $f_G(\delta) = \frac{r^*(c)}{\operatorname{card}(E(G))}$  be the critical edge fraction. Then we define the *fragility* of the graph G as:

$$\mathcal{F}_{\delta}(G) = 1 - \frac{f_G(\delta)}{f_{\text{comp}}(\delta)},\tag{21}$$

where,

$$f_{\text{comp}} = \frac{\operatorname{card}(E(K_n)) - \left(\left\lfloor \frac{n}{c} \right\rfloor \operatorname{card}(E(K_c)) + \operatorname{card}(E(K_b))\right)}{\operatorname{card}(E(K_n))}$$
(22)

is the critical edge fraction of the complete graph.

By the assertion above, any graph G,  $f_{\text{comp}}(\delta) \geq f_G(\delta)$  ( $\forall \delta$ ), which if true means that  $\mathcal{F}_{\delta}(G) \in [0,1]$ . Now that the notion of fragility is defined, it is only natural to examine what it means for a graph to be robust.

### **Definition II.8.** Robust graphs:

We call a graph robust if for a given  $\delta < 1$ ,  $\mathcal{F}_{\delta}(G) < \epsilon$ , where  $0 < \epsilon << 1$ . Additionally we will call a graph asymptotically robust if  $\forall \delta < 1$ ,  $\mathcal{F}_{\delta}(G) \to 0$  when  $n \to \infty$ .

Clearly the complete graph is asymptotically robust, now we will show that CEB graphs are robust as well.

**Theorem II.3.** CEB Graphs Are Asymptotically Robust If a graph is CEB then it is asymptotically robust.

*Proof.* Note that in the case of n even and c even we have:

$$\mathcal{F}_{\delta}(CEB_n) = 1 - \frac{\frac{n^2 - n}{2} \left[ \left( \frac{n}{2} \right)^2 - \left( \left\lfloor \frac{n}{c} \right\rfloor \left( \frac{c}{2} \right)^2 + \left( \frac{b}{2} \right)^2 \right) \right]}{\left( \frac{n}{2} \right)^2 \left[ \left( \frac{n^2 - n}{2} \right) - \left( \left\lfloor \frac{n}{c} \right\rfloor \left( \frac{c^2 - c}{2} \right) + \left( \frac{b^2 - b}{2} \right) \right) \right]}.$$
 (23)

For n even and c odd,

$$\mathcal{F}_{\delta}(CEB_n) = 1 - \frac{\frac{n^2 - n}{2} \left[ \left( \frac{n}{2} \right)^2 - \left( \left\lfloor \frac{n}{c} \right\rfloor \left( \frac{c^2 - 1}{4} \right) + \left( \frac{b^2 - 1}{4} \right) \right) \right]}{\left( \frac{n}{2} \right)^2 \left[ \left( \frac{n^2 - n}{2} \right) - \left( \left\lfloor \frac{n}{c} \right\rfloor \left( \frac{c^2 - c}{2} \right) + \left( \frac{b^2 - b}{2} \right) \right) \right]}, \tag{24}$$

in the case of n odd and c even,

$$\mathcal{F}_{\delta}(CEB_n) = 1 - \frac{\frac{n^2 - n}{2} \left[ \left( \frac{n^2 - 1}{4} \right) - \left( \left\lfloor \frac{n}{c} \right\rfloor \left( \frac{c}{2} \right)^2 + \left( \frac{b^2 - 1}{4} \right) \right) \right]}{\left( \frac{n^2 - 1}{4} \right) \left[ \left( \frac{n^2 - n}{2} \right) - \left( \left\lfloor \frac{n}{c} \right\rfloor \left( \frac{c^2 - c}{2} \right) + \left( \frac{b^2 - b}{2} \right) \right) \right]}, \tag{25}$$

and finally for n odd and c odd,

$$\mathcal{F}_{\delta}(CEB_n) = 1 - \frac{\frac{n^2 - n}{2} \left[ \left( \frac{n^2 - 1}{4} \right) - \left( \left\lfloor \frac{n}{c} \right\rfloor \left( \frac{c^2 - 1}{4} \right) + \left( \frac{b}{2} \right)^2 \right) \right]}{\left( \frac{n^2 - 1}{4} \right) \left[ \left( \frac{n^2 - n}{2} \right) - \left( \left\lfloor \frac{n}{c} \right\rfloor \left( \frac{c^2 - c}{2} \right) + \left( \frac{b^2 - b}{2} \right) \right) \right]}.$$
 (26)

Examining the case of n even and c = n - 1 and noting that for b = 1 the term containing b is 0,

$$\mathcal{F}_{\frac{n-1}{n}}(CEB_n) = 1 - \frac{\frac{n^2 - n}{2} \left[ \left(\frac{n}{2}\right)^2 - \left(\left(\frac{(n-1)^2 - 1}{4}\right)\right) \right]}{\left(\frac{(n-1)^2 - 1}{4}\right)^2 \left[\left(\frac{n^2 - n}{2}\right) - \left(\left(\frac{(n-1)^2 - (n-1)}{2}\right)\right)\right]}.$$
 (27)

Taking the limit of Eq. 27 as  $n \to \infty$  and noting that the largest terms in both the numerator and denominator are  $\frac{1}{8}n^4$  it is easy to see that in this case  $\lim_{n\to\infty} \mathcal{F}_{\frac{n-1}{n}}(\text{CEB}_n) = 0$ . For n even and c = n-2 it can be seen that :

$$\mathcal{F}_{\frac{n-2}{n}}(CEB_n) = 1 - \frac{\frac{n^2 - n}{2} \left[ \left( \frac{n}{2} \right)^2 - \left( \left( \frac{(n-2)}{2} \right)^2 + 1 \right) \right]}{\left( \frac{n}{2} \right)^2 \left[ \left( \frac{n^2 - n}{2} \right) - \left( \left( \frac{(n-2)^2 - (n-2)}{2} \right) + 1 \right) \right]}.$$
 (28)

Eq. 28 again leads to  $\mathcal{F}_{\frac{n-2}{n}}(CEB_n) = 0$  in the limit as  $n \to \infty$ . In general for  $k < \frac{n}{2}$  we have:

$$\mathcal{F}_{\frac{n-k}{n}}(\text{CEB}_n) = \begin{cases} 1 - \frac{\frac{n^2 - n}{2} [(\frac{n}{2})^2 - ((\frac{(n-k)}{2})^2 + (\frac{k}{2})^2)]}{(\frac{n}{2})^2 [(\frac{n^2 - n}{2}) - ((\frac{(n-k)^2 - (n-k)}{2}) + (\frac{k^2 - k}{2}))]}, & \text{for k even} \\ 1 - \frac{\frac{n^2 - n}{2} [(\frac{n}{2})^2 - ((\frac{(n-k)^2 - 1}{4}) + (\frac{k^2 - 1}{2}))]}{(\frac{(n-k)^2 - 1}{2})^2 [(\frac{n^2 - n}{2}) - ((\frac{(n-k)^2 - (n-k)}{2}) + (\frac{k^2 - k}{2}))]} & \text{for k odd.} \end{cases}$$

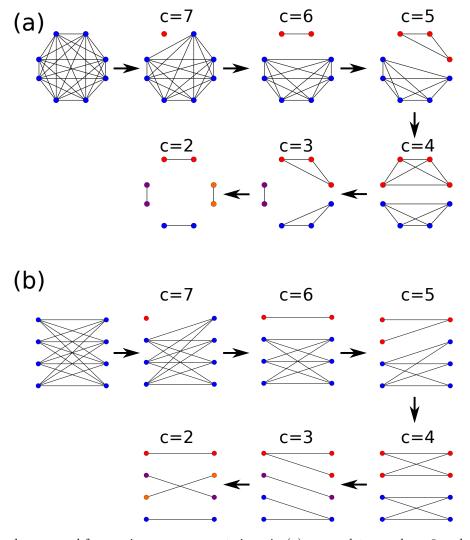


FIG. 3: Minimal edge removal for maximum component size c in (a) a complete graph on 8 nodes ( $K_8$ ) and (b) a CEB graph on 8 nodes (CEB<sub>8</sub>).

Eq. 29 approaches 0 as  $n \to \infty$  for all k. Now since n is assumed even, the case of k = n/2 will be examined, this leads to:

$$\mathcal{F}_{0.5}(CEB_n) = \frac{\frac{n^2 - n}{2} \left[ \left( \frac{n}{2} \right)^2 - 2 \left( \frac{n}{4} \right)^2 \right]}{\left( \frac{n}{2} \right)^2 \left[ \frac{n^2 - n}{2} - 2 \frac{\left( \frac{n}{2} \right)^2 - \frac{n}{2}}{2} \right]},$$
(30)

so we find that  $\mathcal{F}_{0.5}(\text{CEB}_n) = 0$  as  $n \to \infty$ . Since the only thing that changes for increasing k beyond this point is the prefactor in front of the third terms in both the numerator and denominator, it is clear that for all k  $\mathcal{F}_{\frac{n-k}{n}}(\text{CEB}_n) = 0$  as  $n \to \infty$ . This completes the proof for the case of n even. The proof follows similarly when n is odd.

Of note is that this is not true in general for graphs with the same number of edges as the CEB. For instance consider the case of even n with two complete graphs of size  $\frac{n}{2}$  connected together by n edges, which we will call a generalized barbell or GB graph. Such a graph has the same number of edges as the CEB, and yet it is clear for  $\mathcal{F}_{0.5}(GB_n) = 1$  as  $n \to \infty$  since the number of edges of the GB graph grows as order of  $n^2$  but the number of edges required to split the GB graph in half grows as n, as opposed to the complete graph in which both the number of edges and the number of edges required to split it in half grows as order of  $n^2$ .

An example of efficient destruction of both CEB graphs (Fig. 3(b)) and the complete graph (Fig. 3(a)) is shown in Fig. 3. It can be seen that qualitatively both types of graphs fall apart at the same rate.

#### III. METHODS FOR ESTIMATING FRAGILITY

In certain instances, as was the case for the CEB<sub>n</sub> graph, it is possible to obtain a closed form expression for the fragility of a graph. However except for certain special cases, such an expression may be unknown or not exist as is typical for graphs found in the real world. For this reason,  $\mathcal{F}_{\delta}$  must be estimated. In this section we outline a greedy method for the estimation of  $\mathcal{F}_{\delta}$ .

### A. Greedy Removal

In most prior work [18–26] the fragility of a network was estimated by greedy removal of either edges or nodes as given in Algorithm 1. For edge removal, it is typical to apply a "destruction function"  $f(\cdot)$  to each edge and choose the edge which maximizes the destruction of the network. However recently there has been a realization [27] that this may not be an optimal attack strategy, in other words that networks may be more fragile than previously thought. For our purposes we measure the amount of destruction by the size of the LCC after the edge has been removed. A smaller LCC implies a large value for f. A typical metric used to determine which edge to remove at each step is the edge betweenness. Edges with high edge betweenness are generally thought of as being of high importance to the network. Thus the value of f in this case is the edge betweenness. An alternative attack strategy will also be used in this work, one related to the minimum degree node. In this case, every edge attached to the node of minimum degree in the network will have the same value of f, while edges for higher degree nodes have smaller values of f. Therefore the edges of the minimum degree node will be attacked first, until all such edges are stripped away.

```
Algorithm 1: Greedy
EdgeRemoval (G(V,E),r,f(\cdot))
```

```
Data: G(V, E): Graph with vertices V and edge set on n edges, E, where E = \{e_1, e_2, \dots, e_n\}
r: Number of removals
f(\cdot): The "destruction function"

Result: G': Reduced graph

Initialization: set E' = E

for l = 1 : r do

| for k = 1 : n - l + 1 do
| a_k = f(e_k), e_k \in E'
end
| b = \arg\max_k a_k
E' = E'/e_b
end

Return: G'(V, E')
```

Using greedy removal is a computationally efficient method to search for a set of edges  $R \subseteq E$  to be removed from the edge set E of the graph. We must resort to such a strategy in our search because the number possible edge sets for removal grows as r!, where  $r = \operatorname{card}(R)$ . However greedy algorithms are known to produce sub-optimal results in certain circumstances, particularly if they are applied without corrective steps [28–30].

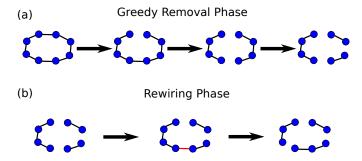


FIG. 4: Greedy Removal with Rewiring. Here we show how the algorithm works on an 8 node cycle with 3 edges to be removed. First in (a) greedy removal is performed, then in (b) the algorithm attempts to rewire edges out of the LCC. If the rewiring results in a reduction in the size of the LCC, then it is accepted. Notice that the red edge in (b) existed originally in the network, thus the rewiring is constrained by the original network topology.

### B. Greedy Removal with Rewiring

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Algorithm 2: RewiringRemoval(G(V, E), r, f(\cdot))

Data: G(V, E): Graph with n vertices V and edge set on m edges, E, where E = \{e_1, e_2, \dots, e_m\}

V = \{v_1, v_2, \dots, v_n\}

V = \{v_1, v_1, v_2, \dots, v_n\}

V = \{v_1, v_2, \dots, v_n\}
```

To better estimate the fragility of a network, we must move beyond a simple greedy algorithm. For this purpose we begin by using two greedy removal strategies as the first stage, one chooses the edge with largest edge betweenness inside the LCC at each step, the other targets the edges of the lowest degree node in the LCC. In the early stages the fastest way to destroy a network is frequently to attack the minimum degree nodes, though this is not always the case. However in later stages, especially for  $\delta << 1$ , the edge betweenness strategy is the most effective targeted attack strategy. Thus combining these two a more optimal set of edge removals may be obtained for any given G and G

After the greedy removal phase has been completed there is a perturbed network G'(r) with edge set E'. A second stage of the algorithm is now performed, which involves rewiring edges from the  $LCC^{(1)} = LCC(G'(r))$  to components outside of  $LCC^{(1)}$ . The rewiring is constrained by the original network structure as shown in Algorithm 2. In this stage candidate nodes are identified from  $LCC^{(1)}$ , with recognition that a node cannot be rewired out of  $LCC^{(1)}$  if it has more edges inside  $LCC^{(1)}$  than edges which have been removed from that node. Thus only a subset of nodes in  $LCC^{(1)}$  are chosen for the attempted rewiring. Once this subset has been determined, nodes from  $LCC^{(1)}$  are rewired to other components of the network, and edges can only be swapped out for edges which were removed from G.

The rewiring algorithm also faces a combinatorial problem. To see that this is the case, let E'' be the edges of LCC<sup>(1)</sup>. Now define

$$G = (V, E),$$

$$G' = (V, E'),$$

$$LCC^{(1)} = (V', E''),$$

$$V' = \{v_{p_1}, ..., v_{p_l}\}$$

$$E = \{e_1, ..., e_m\}$$

$$S_i^{(1)} = \{e_j | e_j \in [(v'_{p_i} \times V) \cup (V \times v'_{p_i})] \cap E\},$$

$$S_i^{(2)} = \{e_j | e_j \in [(v'_{p_i} \times V') \cup (V' \times v'_{p_i})] \cap E''\}$$
(31)

Clearly if

$$\frac{\operatorname{card}(S_i^{(1)})}{2} - \operatorname{card}(S_i^{(2)}) > 0, \tag{32}$$

then there will be choices of which edges to use from the original edge set for rewiring. When this issue arises a single random set of edges of with cardinality  $\operatorname{card}(S_i^2)$  is chosen.

The rewiring is only accepted if the size of LCC(G'(r)) decreases, that is if we let  $LCC^{(2)}$  be the largest connected component of the rewired graph G''(r), then rewiring is only performed if

$$\operatorname{card}(\operatorname{LCC}^{(2)}) < \operatorname{card}(\operatorname{LCC}^{(1)}). \tag{33}$$

It is possible that the rewired graph may allow for additional rewiring, so the algorithm is applied recursively until the largest component no longer decreases in size. This algorithm will never do worse than greedy removal. Once the rewiring stage is completed for both greedy strategies a final stage is completed, as described below. The candidate removal set with the fewest edges removed is then chosen among the two candidate sets, one from the minimum degree attack strategy and the other from edge betweenness.

```
Algorithm 3: IterativeAddBack(G(V, E), r, f(\cdot), c)
```

```
Data: G(V, E): Graph with vertices V and edge set on n edges, E, where E = \{e_1, e_2, \dots, e_n\}
r: Number of removals
f(\cdot): The destruction function
c: The maximum allowed \operatorname{card}(LCC)

Result: G': Final graph

Initialization: set G'(V, E') = \operatorname{RewiringRemoval}(G(V, E), r, f(\cdot)), n = \operatorname{card}(E).
for l = 1 : n do

| if \operatorname{card}(LCC(G' \cup e_l)) \le c then
| E' = E' \cup e_l
| end
| end
| Return: G'(V, E')
```

Algorithm 3 acts as the final step in the new algorithm for estimating the fragility of the network. This stage is performed after the greedy removal and rewiring stages have been completed. It involves iteratively adding back any edges from the original network G to the perturbed graph G' which do not increase the size of the LCC(G') beyond the largest allowable component size c. This final stage allows components (typically other than the LCC but not necessarily after rewiring) to be "regrown" up to cardinality c. Code for this method is made available at [34]

### IV. RESULTS

In this section, a comparison of the various methods outlined above will be presented. Random edge removal and targeted attack are performed, both on synthetic as well as real networks. Performance of these attack methods is examined in terms of the network fragility measure developed above.

### A. Real Network Data

A real network is generated from Safegraph data for comparison of the performance of the techniques. Data was obtained from the Safegraph mobility dataset [31]. Location data was collected from over 20 million devices. We first consider a shopping mall, where each business has an independent entry (see Figure 5 for layout).

The Safegraph-tracked devices in the shopping mall were classified as (a) those whose location is precise to the business they are in; and (b) those devices are in the mall, but their location in the mall is not known more precisely. These data are available sampled at hourly intervals.

To build a network, we combine the Safegraph data with a publicly available layout of the mall. We start with a single snapshot in time. Devices whose locations are known at the business level were first placed. The rest of the devices were placed at random locations in the mall. A network is formed assuming Bluetooth connectivity between devices. Nominally, a ten-meter range is assumed for Bluetooth communications. A pair of devices within this range is assumed to be connected, unless there are walls between them. For each wall between the pair of devices, the range is halved [32]. An example of a network in this way is shown in Figure 6, where the red dots are devices identified to be in specific businesses, and the blue dots, the devices placed at random locations.

Network Type	With rewiring( $<\mathcal{F}_{0.5}>$ )	Min. Degree	Edge Betweennness
ER	0.4294	-0.0315	-0.0112
BA	0.3614	0.0535	-0.0666
WS	0.7395	-0.1012	0.5768
Safegraph Mall	0.8421	0.8421	0.8398

TABLE I: Estimated Network Fragility. These are the estimated fragility values with  $\delta = 0.5$  for three different network types, Erdős-Rényi (ER), Barabasi-Albert (BA) and Watts-Strogatz (WS). The values are averaged over 100 network realizations, and in each run the networks each had 500 nodes and exactly 1984 undirected edges.

## B. Random Edge Removal

We consider three random graph architectures to assess the effects of random edge removal: the Watts-Strogatz (WS) model, the Erdős-Rényi (ER) model, and the Barabási-Albert (BA) model. In addition we consider the Safegraph mall network described above. For the synthetic graphs for each random realization, the degree distribution was determined. Edges were targeted at random, and after each edge was removed, the degree distribution of the resulting graph was evaluated. The process was continued until all edges were removed. The Hellinger divergence between the original graph degree distribution and the degree distribution after each removal was calculated. This entire process was repeated multiple times for each graph configuration and the Hellinger divergence values were averaged across 100 trials for each number of removed edges. Videos were produced, which are included as supplementary material, of the change both in network structure and Helinger divergence as each edge is removed. Random removal is not an efficient attack mechanism however and thus gives poor estimates of the fragility of a network. Indeed, in all cases examined the estimated value of  $\mathcal{F}_{0.5}$  was negative, which clearly makes random removal inappropriate for this estimation.

## C. Targeted Attack

We examined three types of targeted attack, attacking the minimum degree nodes, attacking sequentially via edge-betweenness centrality, and the newly proposed method with rewiring. For comparison with the random edge removal method, the targeted attack methods were applied to the same networks as random removal, until  $E = \emptyset$ . Videos, included as supplementary material, have been produced showing changes in the degree distribution as well as the Hellinger divergence [33] between the initial degree distribution and the new degree distribution (after each edge removal) for the various attack strategies.

In Table I the fragility at  $\delta=0.5$  is estimated (i.e. the largest connected component is no larger than half of all of the nodes) by averaging over 50 realizations of the BA, ER and WS graphs, each with 500 nodes and exactly 1984 edges. For the BA network the model parameters were n=500 and m=4, for the ER graph n=500, and p=500 is chosen for each network realization so that the number of edges is exactly 1984, and for the WS network, n=500 and k=8, p=0.2, and then edges were removed at random until the network had exactly 1984 edges. The final entry is the estimated fragility at  $\delta=0.5$  for the Safegraph mall network shown in Fig. 5 (described below).

In the videos it can be seen that the various attack strategies lead to quite different degree distributions as measured by the Hellinger divergence. To further illustrate this point, in Table I, it is shown that the estimated average value of  $\mathcal{F}_{0.5}$  is lower for both the minimum degree and edge betweenness attack strategies than the estimated

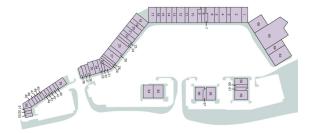


FIG. 5: Layout of the mall. Each numbered space is a business in the shopping mall. All businesses have outside entries, and are not connected to each other internally.

value our method mixing both along with rewiring. This suggests that graphs are generally more fragile to edge removal than was previously understood. Additionally, we note the high fragility of the Safegraph mall network, which suggests that removing a small number of edges in a person-to-person interaction network may quickly disintegrate the network. This may have implications for strategies for limiting epidemic spread among other applications.

#### V. CONCLUSION

In this work we have presented a new measure for the fragility of a network to edge attacks. From this measure, a measure of robustness is derived. The concept of asymptotic robustness is presented. It is shown that in this new measure, the complete graph is robust. Additionally, a class of graphs which is sparser than the complete graph is shown to be asymptotically robust. Finally an algorithm for estimating the fragility of a general graph is presented. It is shown that graphs tend to be more fragile than previous methods would indicate and thus care should be taken when designing networks which may be subject to edge removal.

This work focused on the case in which we have global information about the edges of a graph. Frequently, only local information about the graph structure may be obtained. This suggests that in future work it may be beneficial to estimate the fragility when such global information is unavailable.

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FIG. 6: Devices in the mall are connected to form a network. A pair of devices are connected if they are within communications range of each other. Red dots are devices identified to be in specific businesses, and the blue dots, the devices placed at random locations.

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