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CONTROLLING CHAOS AND THE INVERSE FROBENIUS–PERRON PROBLEM: GLOBAL STABILIZATION OF ARBITRARY INVARIANT MEASURES

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The inverse Frobenius–Perron problem (IFPP) is a global open-loop strategy to control chaos. The goal of our IFPP is to design a dynamical system in $\Re^n$ which is: (1) nearby the original dynamical system, and (2) has a desired invariant density. We reduce the question of stabilizing an arbitrary invariant measure, to the question of a hyperplane intersecting a unit hyperbox; several controllability theorems follow. We present a generalization of Baker maps with an arbitrary grammar and whose FP operator is the required stochastic matrix.

1. Introduction

The realization that the determinism which defines chaos also opens the possibility of controllability, has garnered a great deal of attention and research, in recent years [Chen & Dong, 1998; Kapitaniak, 1996]. Sensitive dependence to initial conditions implies that a small variation in initial conditions or parameter values can potentially lead to rapid and dramatic changes in system output. This motivation has focused chaos control research to take advantage of this sensitivity, by developing control algorithms which achieve their objectives without crude or drastic perturbations to the system. Control objectives are therefore typically sought within the already present and richly varied orbits.

Two main types of controlling chaos objectives are: (1) stabilization of an unstable periodic orbit; typified by the original method due to Ott, Grebogi and Yorke, OGY [Ott *et al.*, 1990]; a local feedback control-loop is applied once a chaotically wandering trajectory has wandered within a small neighborhood of the target, and hence, ergodicity and patience (transients can be long) can be thought of as the global strategy. (2) Targeting solves the next natural (global) question; can we find a very fast trajectory to near the target point [Shinbrot *et al.*, 1993; Kostelich *et al.*, 1993; Bollt & Meiss, 1995; Schweizer & Kennedy 1995; Bollt & Kostelich, 1998]. In both cases, *taking advantage of sensitive dependence to initial conditions* uniquely allows chaos control algorithms to achieve their goals through *small perturbations.* However, these are both closed-loop programs.

In this paper, we pursue a recent alternative method of controlling chaos, based on altering the statistical properties of the attractor. One chooses a dynamical system whose corresponding Frobenius–Perron operator has the targeted density as its "physical" density fixed point. This inverse problem is known as the *Inverse Frobenius–Perron Problem* [Góra & Boyarsky, 1998, 1996, 1993, 1997; Koga, 1991; Grossmann *et al.*, 1997] (or IFPP for short), and we develop here a new approach for its solution. This can be described as a strategy to globally

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stabilize a target state or distribution of the perturbed dynamical system, and unlike the abovementioned control algorithms, IFPP is open-loop. Additionally, we will achieve control with small perturbations of the map. Furthermore, our approach yields simple controllability theorems.

To be more precise, we slightly perturb a dynamical system, for which we assume only an *observed "physical" invariant measure,* to a nearby dynamical system which has the desired statistics. We show that "atypical" atomic invariant measures can be globally selected. Additionally, some noninvariant measures can be achieved by a *bounded away* from zero perturbation.

A major requirement of our matrix-based approach to the IFPP is our ability to pass from the targeted Frobenius–Perron matrix back to an appropriate dynamical system. We call this the "inverse Ulam problem" (IUP). For one-dimensional systems, the IUP is completely solved [Góra & Boyarsky, 1993, 1997] by virtue of the fact that the Ulam conjecture is proven [Li, 1976; Boyarsky & Haddad, 1981]. In higher dimensional dynamical systems, a rigorous footing of Ulam's conjecture is incomplete, except for special cases [Froyland, 1995, 1997; Boyarsky & Lou, 1991; Ding & Zhou, 1995]. In Sec. 5, we have designed a class of piecewise affine and area-preserving dynamical systems generalizing Baker's transformations which: (1) mimic the Anosov expanding/contracting nature of typical maps, (2) have the grammar generated by an arbitrary transition matrix on a Markov partitioning grid, and they can be designed to yield the desired FP operator.

P. Góra and A. Boyarsky have several works related to the IFPP, in particular as it relates to the control of one-dimensional dynamical systems [Góra & Boyarsky, 1998, 1996, 1993, 1997]. In [Góra & Boyarsky, 1996], P. Góra and A. Boyarsky presented an elegant graph theoretic solution; they constructed nearby dynamical systems which maximize the invariant density supported over a so-called target set, selecting cycles of a digraph representation which, frequently hit the targeted set(node). In [Góra & Boyarsky, 1998, 1997], the authors offer an algorithm to dynamically remove an interval, and its pre-iterates, from the density support for a sequence of approximating one-dimensional maps.

Two other approaches to the IFPP can be found in the literature, both of which require large perturbations to the map. These are: (1) Integration of the Frobenius–Perron operator (of a 1-D map), with assumptions on the map's form to force a unique solution (e.g. assume a symmetric unimodal map), can be made to yield a 1-D differential equation whose solution is the map, and can be found in closed form for certain special target distributions. See for example [Koga, 1991], (2) Conjugation transformation to conjugate f to a piecewise-linear map g, $f \circ h = h \circ g$, for a g with a uniform invariant density, such that the conjugacy composed with the density gives the density of f [Grossmann *et al.*, 1977; Mori *et al.*, 1981; Baranovsky & Daems, 1995].

2. Motivation and Definitions

Consider the dynamical system,

$$f : M \to M, \quad M \subset \Re^m . \tag{1}$$

Our objective is to control the "physical" invariant measure. Given a measure space (M, A, μ), then μ is f-invariant if , $\mu(S) = \mu(f^{-1}(S))$, for all Borel sets $S \in A$ where f^{-1} denotes the (perhaps many-to-one) preimage. In general, there are infinitely many invariant probability measures for a chaotic f, and we wish to alter the *typically observed* invariant measure.

The so-called Sinai–Bowen–Ruelle (SBR) measure, when it exists, is widely considered to be the "physical measure" of the attractor [Bowen & Ruelle, 1975], and it is this measure which one (almost always) observes in computer simulations. The question of existence of an SBR is open in the general case [Ott, 1994], although there are existence results for some smooth one-dimensional maps. For Axiom A diffeomorphisms, as well as Anosov diffeomorphisms, μ_{SRB} is widely considered to be the physical measure, but these special cases do not include many common nonuniformly hyperbolic examples. Recently however, a μ_{SRB} has been established for certain parameter values of the Hénon maps [Benedicks & Carleson, 1991; Benedicks & Young, 1993]. Numerically however, the existence of physical measures seems to be quite prevalent. In fact, for the purposes of this paper we do not require that the original function f can be proven to have a natural measure; we wish only to construct a nearby function approximate $f + \delta f$, which has the desired statistics.

The *Frobenius–Perron* (FP) operator is central to discussions of invariant measures as any invariant density is a fixed point of the operator, $P_f\rho(x) = \rho(x)$. Intuitively, this operator maps densities to densities under the action of the map. More precisely [Lasota & Mackey, 1997], the FP operator P_f associated with a DS Eq. (1) is itself a dynamical system on $L^1(M)$ disributions, $P_f : L^1(M) \to L^1(M)$, $\rho \in L^1(M)$, according to, $\int_B P_f\rho(x)d\mu(x) = \int_{f^{-1}(B)} \rho(x)d\mu(x)$, for all Borel sets $B \in \mathcal{B}$. When the derivatives are nonsingular, we may write [Lasota & Mackey, 1997],

$$P_f\rho(x) = \sum_{\{y:f(y)=x\}} \frac{\rho(y)}{|Df(y)|}, \tag{2}$$

where $|Df(y)|$ is the determinant of the Jacobian derivative of f, at y.

An important simplification when approximating P_f on an equipartition (call the grid cells Q_i) of the state-space M, comes from application of the so-called Ulam method conjectured by S. Ulam [Ulam, 1960], which projects the action of P_f on the infinite dimensional space $L^1(M)$ onto a finite element linear subspace, $A : \Delta_n \to \Delta_n$, where Δ_n is given by characteristic functions on the grid cells $\{Q_i\}_i$. The stochastic matrix A is defined,

$$A_{i,j} = \frac{m(Q_j \cap f^{-1}(Q_i))}{m(Q_j)}, \tag{3}$$

which gives the fraction of Q_j which maps to Q_i, $m(\cdot)$ denotes the Lebesgue measure on M, allowing f^{-1} to have multiple branches. Ulam conjectured that the matrix A is a good approximation to P_f in the following sense. A sequence of refining grids over the state space M give a sequence of stochastic matrices A, with a sequence of dominant fixed eigenvectors $A \cdot \mathbf{v} = \mathbf{v}$, and these eigenvectors converge weakly to the dominant eigendensity, $P_f\rho(x) = \rho(x)$, as the grid is refined.

The Ulam "conjecture" was proven by T. Y. Li [1976] using bounded variation arguments, requiring that $f \in$ piecewise $C^2[0, 1]$, with $M > \inf|f'| > 2$. The n-dimensional generalization was proven by G. Froyland in the cases of a Markov partition of an expanding Anosov diffeomorphism, using symbol dynamics techniques in [Froyland, 1975], and by mixing arguments in [Froyland, to appear]. The conjecture was also proven by Boyarsky and Lou [1991], for expanding Jablonski transformations (n-dimensional maps such that, on a grid square, each component-wise function of f is a function of only one independant coordinate variable x_i), again using bounded-variation arguments modeled after the methods in [Li, 1976]. Likewise the methods in [Ding & Zhou, 1995] for piecewise-affine functions also use bounded variation arguments and therefore require an expanding transformation [Proppe *et al.*, 1990].

In practice, using a finite length test orbit $\{x_j\}$, we can approximate the action of the map on discrete densities, alternate to Eq. (3), by developing the transition matrix A,

$$A_{i,j} = \frac{\#\{x_k \text{ such that } x_k \in Q_j \text{ and } f(x_k) \in Q_i\}}{\#\{x_k \in Q_j\}}, \tag{4}$$

by normalizing A so that each column sums to 1, i.e. the total probability of the transition from any given state to all other states is 1.

3. The Control Problem and the Euclidean Norm Solution

Given the dynamical system Eq. (1), which may or may not have an invariant SBR measure μ (we do not require one), with corresponding probability density function ρ, and given another arbitrary probability distribution function $\rho + \delta\rho$, we wish to find a nearby (in the sup-norm sense) dynamical system $f + \delta f$,

$$f + \delta f : M \to M, \quad M \subset \Re^m, \tag{5}$$

such that this dynamical system has an SBR invariant measure $\mu + \delta\mu$ with the desired probability density function $\rho + \delta\rho$. This will not be possible for all $\rho + \delta\rho$, but we discuss controllability in the next section. Note that our targeted transformations will typically be discontinuous.

We consider a sequence of dynamical systems f_n which sup-norm approach f, such that f_n has exactly the Frobenius–Perron matrices A_n, and therefore their invariant measures μ_n have densities ρ_n which follow the sequence of dominant eigenvectors $\mathbf{v}_n$. In this section we show the linear algebraic construction of A_n, and in Sec. 5 we relate a transformation f_n for each A_n, which we call the IUP.

In terms of matrices, the control problem is stated as follows: Given a stochastic matrix A with stationary eigenvector $\mathbf{v}$, we wish to perturb A so that we achieve a desired stationary eigenvector $\mathbf{v} + \delta\mathbf{v}$

$$(A + \delta A) \cdot (\mathbf{v} + \delta\mathbf{v}) = \mathbf{v} + \delta\mathbf{v}. \tag{6}$$

In general, we do not expect that this is a well-defined inverse problem. There are infinitely many matrices δA likely which satisfy Eq. (6) for given data A, $\mathbf{v}$, and $\mathbf{v}+\delta\mathbf{v}$. However, we intend to do this *within* the grammar of A, which allows us to closely approximate f. This motivates the following objectives of the linear algebraic problem:

Control Rules:

1. $(A+\delta A)\cdot(\mathbf{v}+\delta\mathbf{v}) = \mathbf{v}+\delta\mathbf{v}$.
2. $(A+\delta A)$ is stochastic. Therefore,
   (a) $\sum_{i=1}^{q}(A+\delta A)_{i,j} = 1$, for all j,
   (b) and $0 \le (A+\delta A)_{i,j} \le 1$ for all $(i,\, j)$.
3. Preserve grammar in the sense that we require $\delta A_{i,j} = 0$ for each $(i,\, j)$ that $A_{i,j} = 0$.
4. $\max_{i,j} |\delta A_{i,j}|$ is as small as possible.

Rules 2–4 serve as constraints on Eq. (6). Actually, we will only solve constraints 2(a), and we use 2(b) as a *posterior* condition for evaluating the success of a given solution.

Equation (6) can be expanded, and cancellation of the original term $A\cdot\mathbf{v} = \mathbf{v}$ leaves,

$$\delta A\cdot(\mathbf{v}+\delta\mathbf{v}) = (I-A)\cdot\delta\mathbf{v}\,. \tag{7}$$

Recall that we consider A, $\mathbf{v}$, and $\delta\mathbf{v}$ as the known data, and therefore we rewrite this equation to emphasize the known and unknown components:

$$\delta A\cdot\mathbf{x} = \mathbf{y}\,, \tag{8}$$

$$\text{where,}\ \ \mathbf{x} = \mathbf{v}+\delta\mathbf{v},\ \ \text{and,}\ \ \mathbf{y} = (I-A)\cdot\delta\mathbf{v}\,. \tag{9}$$

Note that the unknowns of the equation are the q^2 entries of the $q\times q$ perturbation matrix $\delta A_{i,j}$. This is a highly under-determined linear system, and the linear equation is a co-dimension-q restriction. From constraint 2(a) follows that δA must column sum to zero, $\sum_{i=0}^{q}\delta A_{i,j} = 0$, which is a further co-dimension-q restriction on the $\delta A_{i,j}$ space of variables, leaving a q^2-2q solution hyperplane.

The grammatical rule 3 further reduces significantly the number of variables. Typically, there are enough constraint 3 equations $\delta A_{i,j} = 0$, to reduce the dimension of the solution hyperplane to just a few degrees of freedom, as made clear in the example following Eq. (10). Suppose that there are T nonzero $\delta A_{i,j}$, then we may rename these variables in the lower case $\{\delta a_i\}_{i=1}^{T}$ as a T-vector of variables, $\boldsymbol{\delta}\mathbf{a}$.

For example, we show a 3×3 matrix δA, in which there are $T = 7$ nonzero entries,

$$\delta A = \begin{pmatrix} da_1 & 0 & da_5 \\ da_2 & 0 & da_6 \\ da_3 & da_4 & da_7 \end{pmatrix}, \tag{10}$$

and we explicitly see the role of the 7-vector variable $\boldsymbol{\delta}\mathbf{a}$ of nonzero $\delta A_{i,j}$ entries which yields,

$$\begin{aligned}\delta A\cdot\mathbf{x} &= \begin{pmatrix} da_1 & 0 & da_5 \\ da_2 & 0 & da_6 \\ da_3 & da_4 & da_7 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 da_1 + x_3 da_5 \\ x_1 da_2 + x_3 da_6 \\ x_1 da_3 + x_2 da_4 + x_3 da_7 \end{pmatrix}.\end{aligned} \tag{11}$$

Reversing these equations so as to emphasize the subordinate role of the 7×1 vector $\boldsymbol{\delta}\mathbf{a}$ of unknown nonzero entries of δA, we get a $2q\times T = 6\times 7$ matrix D,

$$D\cdot\boldsymbol{\delta}\mathbf{a} \equiv \begin{pmatrix} x_1 & 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 & x_3 & 0 \\ 0 & 0 & x_1 & x_2 & 0 & 0 & x_3 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\cdot\,\boldsymbol{\delta}\mathbf{a} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv \mathbf{z}\,. \tag{12}$$

Typically, these hyperplanes are under-determined. Equation (12) defines an infinite solution space; the 7-vector variable $\boldsymbol{\delta}\mathbf{a}$ makes a codimension-6 restriction, leaving a one-dimensional solution space.

We resolve the undetermined solution problem with control objective 4. For ease of calculation, we interpret constraint 4 in terms of minimizing the l^2 norm, but we show that l^∞ minimization also has relevance. Without loss of generality, we minimize the function,

$$n(\boldsymbol{\delta}\mathbf{a}) \equiv \|\boldsymbol{\delta}\mathbf{a}\|_{l^2}^2 = \sum_{i=1}^{T}\delta a_i^2\,. \tag{13}$$

In fact, the solution of Eq. (13) is the well-known least-squares problem, and a particularly stable numerical solution is found using the Penrose pseudoinverse, as calculated by Singular Value Decomposition [Golub & Van Loan, 1989],

$$\boldsymbol{\delta}\mathbf{a} = D^+\mathbf{z}, \quad \text{where,} \quad D^+ = V \cdot \Omega^{-1} \cdot U^t\,, \qquad (14)$$

given the SVD of D, $D = U \cdot \Omega \cdot V^t$. The SVD has numerous nice properties [Golub & Van Loan, 1989; Press *et al.*, 1992], including that $\Omega = \mathrm{diag}(\omega_i)$ is the diagonal matrix of "singular values," and is therefore trivial to invert, and U and V are orthogonal matrices, and are therefore also trivial to invert, $U^{-1} = U^t$, $V^{-1} = V^t$. Most important to us is the property that the SVD solution of an undetermined system automatically selects the solution on the hyperplane which is minimal in the sense that $n(\boldsymbol{\delta}\mathbf{a}) \equiv \|\boldsymbol{\delta}\mathbf{a}\|_{l^2}^2$ is minimized [Golub & Van Loan, 1989; Press *et al.*, 1992].

Note that $\boldsymbol{\delta}\mathbf{a}$, the solution of Eq. (13), is not necessarily a reasonable solution to the control problem conditions 1-4; $\boldsymbol{\delta}\mathbf{a}$ may not give a stochastic $A + \delta A$.

4. Controllability

In the previous section, we gave an algorithm to solve a linear algebraic analog of the IFPP, culminating in Eq. (14). However, we do not expect a reasonable solution for all choices of the vectors $\mathbf{x}$ and $\mathbf{y}$, i.e. we do not expect that an arbitrary A, with fixed grammar, can be controlled to an arbitrary equilibrium distribution, $\mathbf{v} + \delta\mathbf{v}$.

The solution Eq. (14) only solves the control problem constraints 1, 2a, 3, 4; we did not yet consider 2b which we now incorporate as an *ex post facto* controllability requirement. If the solution Eq. (14) yields a reasonable stochastic matrix $A + \delta A$, then 2b says that the solution vector $\boldsymbol{\delta}\mathbf{a}$ must have all of its component entries bounded,

$$0 \le a_k + \delta a_k \le 1\,, \qquad (15)$$

where a_k denotes the $A_{i,j}$, a nonzero entry of A, indexed to match δa_k with the same position in $\delta A_{i,j}$. For an arbitrary A and target distribution $\mathbf{v} + \delta\mathbf{v}$, we can always construct a solution $A + \delta A$ according to Eq. (14), but this solution is not necessarily a stochastic matrix. Another solution, which does not violate condition 2b is possible. The problem is due to the fact that we have minimized $\|\boldsymbol{\delta}\mathbf{a}\|_{l^2}$, in whose topology, "balls" are hyperspheres which do not fit properly into the boxes of Eq. (15); a round peg does not fit into a square hole. The following theorem addresses this issue.

Theorem 1. *If the solution* $\boldsymbol{\delta}\mathbf{a} = D^+\mathbf{z}$ *to control problem 1–4 has a bad component,* $a_k + \delta a_k < 0$ *or* $a_k + \delta a_k > 1$, *then the minimal* $\|\boldsymbol{\delta}\mathbf{a}\|_{l^2}$ *does not yield a feasible solution, but there may be another feasible* $\boldsymbol{\delta}\mathbf{a}$. *However, there is a very bad component if* $(a_k + \delta a_k - 1/2) > \sqrt{T}/2$, *and then there exists no stochastic* $A + \delta A$ *control solution.*

Proof. $D \cdot \boldsymbol{\delta}\mathbf{a} = \mathbf{z}$ defines a $2q$-dimensional hyperplane in $\Re^T$, $T > 2q$. The condition $0 \le a_k + \delta a_k \le 1$, $1 \le k \le T$ defines a compact hyperbox, in $\Re^T$. If the hyperbox intersects the hyperplane, then the "reasonable" solution exists, satisfying conditions 1–4. The center of the box is the point $\mathbf{p} \in \Re^T$, $p_k = 1/2$, and a corner of the hyperbox is a Euclidean distance $\sqrt{T}/2$ from the center, but there are points in the ball which are not in the box if $(a_k + \delta a_k - 1/2) < \sqrt{T}/2$. Figure 1 clearly illustrates the existence of such points. ■

This indicates that to ensure nonexistence of a solution of $D \cdot \boldsymbol{\delta}\mathbf{a} = \mathbf{z}$ in the hyperbox, we must

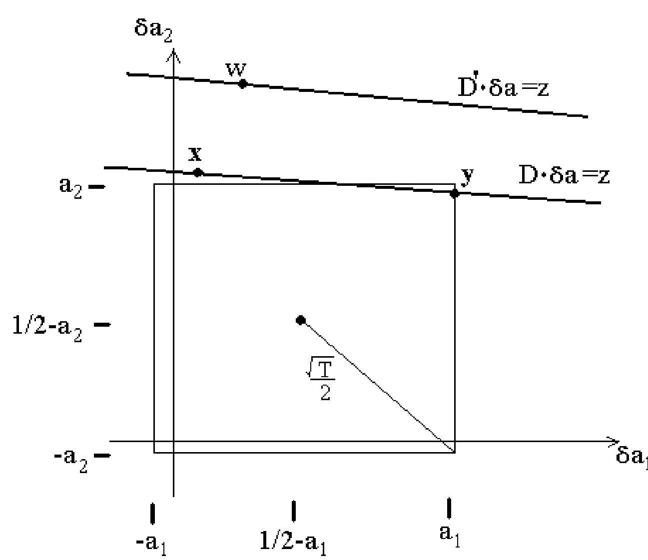

Fig. 1. A low-dimensional characature of the hyperplane $D \cdot \boldsymbol{\delta}\mathbf{a} = \mathbf{z}$ piercing (or not piercing) the stochastic hyperbox $0 \le a_i + \delta a_i \le 1$, $i = 1, 2, \ldots, T$, in $\Re^T$. The point $\mathbf{x}$ is the l^2 norm closest point, on the hyperplane, to the origin, but this point is not in the box. See Theorem 1. However, in this illustration, the hyperplane does pierce the box, and the point $\mathbf{y}$. The l^∞ minimum definitively decides whether $D \cdot \boldsymbol{\delta}\mathbf{a} = \mathbf{z}$ pierces the box; see Theorem 2.

consider the sup-norm, whose "balls" are actually boxes [Kolomogorov & Fomin, 1970]. We state an existence theorem, based on the natural sup-norm of a matrix. Recall that, given an n-vector $\mathbf{x}$, $\|\mathbf{x}\|_\infty \equiv \max_i |x_i|$, and given an $m \times n$ matrix A, the natural sup-norm is $\|A\|_\infty \equiv \sup_{\|\mathbf{u}\|_\infty=1} \|A \cdot \mathbf{u}\|_\infty = \sup_{\|\mathbf{u}\|_\infty \le 1} \|A \cdot \mathbf{u}\|_\infty$ which measures roughly the maximal "stretch" a matrix can affect on a vector space. This matrix norm is easily computed as the maximal absolute row sum [Golub & Van Loan, 1989],

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |A_{i,j}| . \tag{16}$$

Lemma 1. *Given the undetermined linear equation* $A \cdot \mathbf{x} = \mathbf{b}$, *there exists a solution in the unit box,* $-1 \le x_i \le 1$ iff *given the companion system* $\tilde{A} \cdot \tilde{\mathbf{x}} = \tilde{\mathbf{b}}$, (*where* $\tilde{A} = \mathrm{diag}(1/b_i) \cdot A$, *and* $\tilde{\mathbf{b}} = \mathrm{diag}(1/b_i) \cdot \mathbf{b}$, *assuming that* $b_i \ne 0$ *for all* i), *that* $\|\tilde{A}\|_\infty \ge 1$.

Proof. Given the normalized companion system $\tilde{A} \cdot \tilde{\mathbf{x}} = \tilde{\mathbf{b}}$, the definition of $\|\tilde{A}\|_\infty$ considers whether a unit vector (which can be thought of as a candidate solution $\tilde{\mathbf{x}}$ from the unit box) can be "stretched" to attain $\|\mathbf{b}\|_\infty = 1$. ■

Lemma 2. *The undetermined linear system* $A \cdot \mathbf{x} = \mathbf{b}$, *has a solution in the box* $\alpha_i \le x_i \le \beta_i$, *iff a linearly scaled companion system* $A^* \cdot \mathbf{x}^* = \mathbf{b}^*$ *has a solution* $\|\mathbf{x}^*\|_\infty \le 1$.

Proof. Rescale the unit box as follows: $x_i^* = \sigma_i(x_i - \alpha_i) - 1$, where $\sigma_i = 2/(\beta_i - \alpha_i)$ is arrived at by constructing the line through $(\alpha_i, -1)$, and $(\beta_i, 1)$. Hence, $x_i = x_i^*/\sigma_i + (1/\sigma_i + \alpha_i)$ gives $-1 \le x_i^* \le 1$ when $\alpha_i \le x_i \le \beta_i$. Therefore, $\mathbf{x} = \mathrm{diag}(1/\sigma_i) \cdot \mathbf{x}^* + \mathbf{s}$, where $\mathbf{s}_i = (1/\sigma_i + \alpha_i)$, and $A^* = A \cdot \mathrm{diag}(1/\sigma_i)$, and $\mathbf{b}^* = \mathbf{b} - A \cdot \mathbf{s}$. ■

Theorem 2. *Given the stochastic matrix* A, *with stationary distribution* $\mathbf{v}$, *then the target stationary distribution* $\mathbf{v} + \delta\mathbf{v}$ *can be attained within the grammar of* A, *according to control objectives 1–4, iff there exists a solution to the undetermined linear system* $D \cdot \boldsymbol{\delta}\mathbf{a} = \mathbf{z}$, *within the box* $-a_k \le \delta a_k \le 1 - a_k$. *This system has a solution iff a companion system* $\tilde{D}^* \cdot \tilde{\boldsymbol{\delta}}\mathbf{a}^* = \tilde{\mathbf{z}}^*$, *has the matrix natural sup-norm bounded,* $\|\tilde{D}^*\|_\infty > 1$.

Proof. The theorem follows immediately the "twiddle" and "star" operations defined in Lemmas 1 and 2. ■

The Euclidean norm is often the most reasonable interpretation of the word "minimal" in constraint 4, and we showed that this quantity is easily computable by the well established and robust SVD algorithm. However, Theorem 1 shows that the l^2 minimum is not always sufficient to decide nonexistence of a solution, but Theorem 2 indicates that nonexistence of a good (stochastic) l^∞ minimal solution is sufficient. If Theorem 2 indicates that an l^∞ solution does indeed exist, we replace the objective function, Eq. (13), with the infinity-norm.

$$n(\boldsymbol{\delta}\mathbf{a}) = \max_i |\delta a_i|, \quad \boldsymbol{\delta}\mathbf{a} \in H , \tag{17}$$

where H is the set of $\boldsymbol{\delta}\mathbf{a}$ satisfying control rules 1–4 including constraints Eq. (15). This constrained optimization problem is not difficult to solve by repeated linear programming on each δa_i [Press *et al.*, 1992], selecting the maximal–minimized coordinate δa_i, the details of which can be found in a longer preprint at this author's website. The point is, given the l^∞ solution, we can state the following sharp nonexistence theorem, which does not hold for the l^2 solution.

Theorem 3. *Given the stochastic matrix* A, *and target distribution* $\mathbf{v} + \delta\mathbf{v}$, *then if the control rules 1–4 constrained* l^∞ *minimal solution of* $D \cdot \boldsymbol{\delta}\mathbf{a} = \mathbf{z}$, *found by repeated linear programming, yields an* $A + \delta A$ *which is not stochastic, then no such* $A + \delta A$ *exists.*

5. A Piecewise-Affine Transformation for the Given Stochastic Matrix

Given a dynamical system $f : M \to M$, we can easily approximate the Frobenius–Perron operator, as a stochastic matrix, simply by box counting, and keeping track of relative frequencies of transitions between boxes, as already described in Sec. 2, Eq. (4). In the last two sections, we used this stochastic matrix A as a starting point to find a "grammatically nearby" stochastic matrix $A + \delta A$ satisfying control rules 1–4, with a desirable stationary probability distribution. The purpose of this section is to describe the construction of a piecewise-affine dynamical system $f + \delta f$ which: (1) is sup-norm-nearby f, (2) the matrix $A + \delta A$ is *exactly* the Frobenius–Perron matrix for $f+\delta f$, and the ε-grid is a Markov partition. We have already called this second part the inverse Ulam problem (IUP).

5.1. *1-D case*

The argument for the 1-D IUP is as follows. A piecewise-linear function can be found arbitraily (sup-norm) close to any continuous function. The FP operator in Eq. (2) has divisions by the derivative, and a piecewise-linear function has piecewise-constant derivative. So, given a stochastic matrix A and a grid Q_i, we can construct the piecewise-linear function f_n. A typical piecewise-linear function, defined on a grid square Q_i is shown in Fig. 2; construction of each linear segment of f_n easily follows, the key observation being that the slopes of each segment $[l_{i,k}, l_{i,k+1}]$ come from the transition weights $A_{i,j}$,

$$m_{i,k,j} = \pm\frac{\Delta x_j}{l_{i,k+1} - l_{i,k}}\,, \tag{18}$$

by placing intra-grid points $l_{i,k}$ so that the resulting slopes match according to the equation $A_{i,j} = 1/|f_n'(x)|$ and Eq. (2). Details can be found in [Góra & Boyarsky, 1993, 1997].

The grid Q_i is a *Markov partition* of this interval map f_n, defined as follows. Suppose M is an interval, and f_n is a piecewise-linear map, with constant slope s_i for each $P_i = [a_i, b_i]$ of a grid-partition $Q = \cup_{i=1}^{q} Q_i = M$, $\text{int}(Q_i) \cap \text{int}(Q_j) = \emptyset$ if $i \neq j$, *and* Q_i exactly maps onto a union of subintervals, $f(Q_i) = \cup_{k=1}^{l} Q_{j_k}$, $l \leq q$, then the partition is *Markov* and the action of the map on densities is given by a stochastic matrix A [Alligood *et al.*, 1997]. By construction, A is a stochastic matrix. In particular, the dominant eigenvector of A is stationary probability density of f_n, and when f is expansive ($|f'(x)| > 1$ for $x \in \text{int}(Q_i)$) then ρ is constant on each subinterval Q_i of the partition [Boyarsky & Haddad, 1981]. This slope condition can be weakened to $|f'(x)| \geq 1$ if the transition matrix is irreducible and aperiodic.

Since, by construction f_n differs from f by at most the size of a grid square, we have that the sup-norm is bounded in terms of the grid, because we control the approximation by refining the grid,

$$\sup_{x\in I} |f_n(x) - f(x)| \leq \max_i \Delta x_i = \varepsilon\,. \tag{19}$$

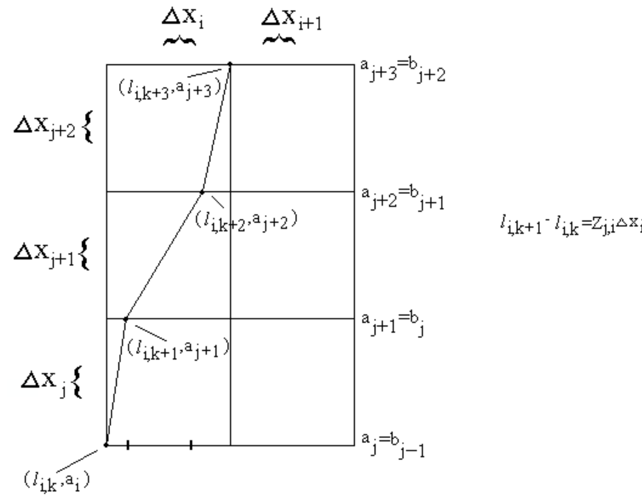

Fig. 2. The piecewise-linear approximate function $f_n(x) : I \to I$, designed for a stochastic matrix A. We show a segment of $f_n(x)$ defined over the line segment (a_i, b_i), with width $\Delta x_i = b_i - a_i$. The interval (a_i, b_i) is divided proportionally by the "contiguous" transition probabilities (three are shown), $A_{j,i}$, $A_{j+1,i}$, and $A_{j+2,i}$. The widths are defined $l_{i,k+1} - l_{i,k} = A_{j,i}\Delta x_i$, $l_{i,k+2} - l_{i,k+1} = A_{j+1,i}\Delta x_i$, and $l_{i,k+3} - l_{i,k+2} = A_{j+2,i}\Delta x_i$.

5.2. *2-D case*

To solve the 2-D IUP, we introduce piecewise-affine transformations $f_n : Q \to Q$, $Q \in \Re^2$, which have A as its Frobenius–Perron operator, on the grid. We designed these transformations to mimic the expanding/contracting and invertible nature of an Anosov diffeomorphism, and long term statistics on the grid Q are the same as A has on its linear space. Furthermore, our construction implies agreement between f and f_n on the grid,

$$\|f(x, y) - f_n(x, y)\|_{\sup(\cup_i Q_i)} \leq \varepsilon\,, \tag{20}$$

again follows by the "grammar-preservation" rule 3. These piecewise-affine transformations can be thought of as a generalization of the Baker's transformations.

Equation (2) suggests that the Frobenius–Perron operator reduces to a matrix when the Jacobian is piecewise constant, *on a Markov partition.* The construction follows that the determinant of the inverse Jacobian matrix describes how the area of a rectangle, in the tangent space, stretches under the action of the inverse transformation. Literally, this means that a rectangle is scaled back to a rectangle if the transformation is piecewise affine on rectangular regions.

Our construction is characatured in Fig. 3, illustrating $100 \cdot A_{j,i}\%$ of the grid-cell Q_i (in gray)

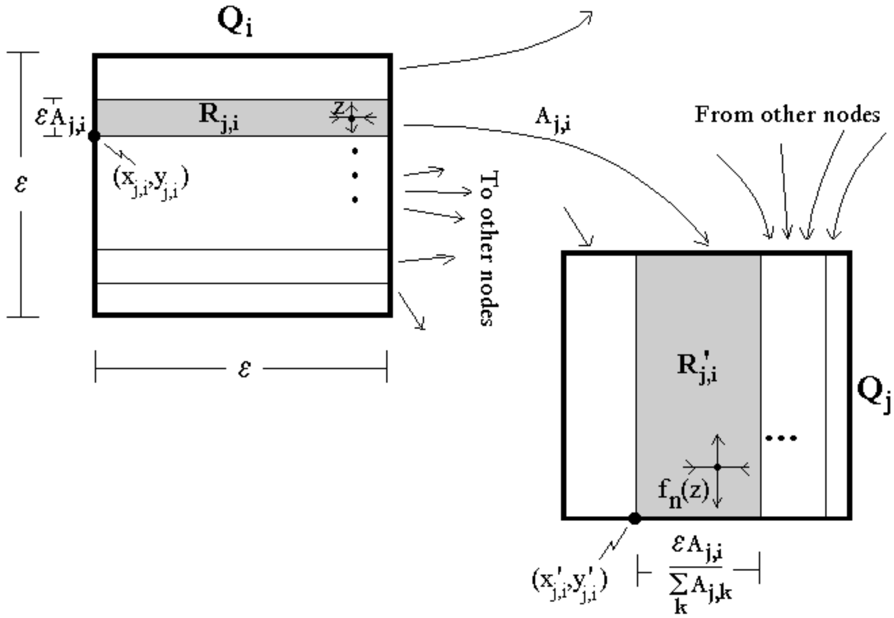

Fig. 3. The piecewise-affine approximate function $f_n(x) : M \to M$, $M \in \Re^2$, in Eq. (21), constructed for a given stochastic matrix A, and designed with Anosov-like expanding and contracting directions.

which maps surjectively onto $100 \cdot A_{j,i}/\sum_k A_{j,k}\%$ of the cell Q_j. This is done by the affine transformation,

$$\mathbf{f}_n^{i,j}(x, y) = \begin{pmatrix} f_n^{j,i}(x) \\ g_n^{j,i}(y) \end{pmatrix} = \begin{pmatrix} \frac{\Delta x'_{j,i}}{\Delta x_{j,i}} & 0 \\ 0 & \frac{\Delta y'_{j,i}}{\Delta y_{j,i}} \end{pmatrix} \cdot \begin{pmatrix} x - x_{j,i} \\ y - y_{j,i} \end{pmatrix} + \begin{pmatrix} x'_{j,i} \\ y'_{j,i} \end{pmatrix} \tag{21}$$

which scales the $\Delta x_{j,i} \times \Delta y_{j,i}$ rectangle $R_{j,i}$, whose lower left corner is $(x_{j,i}, y_{j,i})$, linearly onto the $\Delta x'_{j,i} \times \Delta y'_{j,i}$ rectangle $R'_{j,i}$, whose lower left corner is $(x'_{j,i}, y'_{j,i})$. Our notation convention is that $R_{j,i} \subset Q_i$ is the rectangle in Q_i which maps *into* Q_j, but *onto* $R'_{j,i} \subset Q_j$, where $R'_{j,i}$ is the part of Q_j that came from Q_i. The grid cell Q_i is similarly filled with $R_{k,i}$, which form a grid cover, $(\cup_k R_{k,i} = Q_i$ and $\text{int}(R_{k,i}) \cap \text{int}(R_{l,i}) = \emptyset$ if $k \neq l)$, and likewise Q_j is grid covered by rectangles $R'_{j,m}$ which came from various cells Q_m. In other words, $R_{j,i} \equiv Q_i \cap f_n^{-1}(Q_j)$ and $R'_{j,i} \equiv f(Q_i) \cap Q_j$. For the $\varepsilon \times \varepsilon$ disjoint and rectangular cell cover Q_i, $M \subset \cup_i Q_i$, the "from node" rectangle $R_{j,i} \subset Q_i$ has dimensions determined by the weight $A_{j,i}$ on the arc from Q_i, $(\Delta x_{j,i}, \Delta y_{j,i}) = (\varepsilon A_{j,i}, \varepsilon)$, so that $R_{j,i}$ is a subrectangle with $100 \cdot A_{j,i}\%$ of the Lebesgue measure of Q_i. The lower left edge is defined cumulatively, and based on the grid, $y_{j,i} = \varepsilon \cdot \sum_j A_{j,i}$, and $x_{j,i}$ is the left edge of Q_i. See Fig. 3. The "to node" rectangle $R'_{j,i}$ is defined in a similiar manner, but with a key difference: the total sum of weights on all of the arcs into Q_j need not sum to 1.

We now state several lemmas concerning these piecewise-affine transformations. The proofs are simple and were omitted for reasons of brevity, but they can be found in an earlier version of this manuscript on the author's website.

Lemma 3. *We define the set* $\Psi = \cup_k Q_k - \cup_k[\cup_{l=-\infty}^{\infty} f_n^l(\partial Q_k)]$, *where* ∂Q_k *denotes the boundary of the cell* Q_k. *The transformation* $f_n : \Psi \to \Psi$ *is one-to-one and onto.*

Lemma 4. *The transformation* $f_n : \Psi \to \Psi$ *is hyperbolic if there is no* i, j *such that* $A_{i,j} = 1$, *or equivalently, there are no rows and columns of* A *with fewer than two nonzero entries.*

Lemma 5. *The transformation* $f_n : \Psi \to \Psi$ *is hyperbolic if* A *is irreducible and aperiodic.*

Lemma 6. *The grid* $\cup_i Q_i$ *is a Markov partition when* $f_n : \Psi \to \Psi$ *is hyperbolic.*

In the Examples section, we present numerical evidence to support the following conjecture.

Conjecture 1. *Let $f_n : \cup_i Q_i \to \cup_i Q_i$, defined by Eq. (21), be hyperbolic on $\Psi \subset \cup_i Q_i$, and with Markov partition $\cup_i Q_i$. The stochastic matrix defined, $A_{j,i} = m(f_n^{-1}(Q_j) \cap Q_i)/m(Q_i)$, has a dominant eigenvector* **p**, *corresponding to the eigenvalue $\lambda = 1$, such that the measure μ defined, $\mu(E) = \sum_k m(E \cap Q_k)/m(Q_k) \cdot p_i$, for any Borel set E, coincides with μ_{SBR}.*

Note that μ_{SRB} depends only on rates at which volumes grow, along unstable directions, according to the determinant of the Jacobian projected onto the unstable subspace, $|DT(z)|_{E^u(z)}$. Hence, rates volume decrease along stable manifolds are unimportant to the asymptotic stability of the forward Markov chain. This leaves a great deal of latitude when choosing $\Delta x_{j,i}$, as long as $|DT(z)|_{E^s(z)}$ is chosen to be nonsingular.

6. A Sequence of Discrete Approximates to the Continuous Dynamical System

In this section, we extend a theorem concerning convergence and attainable target distributions, found Góra and Boyarsky [1998]. We also discuss the implications for which discrete approximations of the continuous control problem are expected to have solutions as outlined in the previous sections.

Given a transformation of an interval, $M = [a, b]$, Góra and Boyarsky [1998], showed that not all target measures can be attained by perturbing the dynamical system f, Eq. (1), in an arbitrarily small fashion. We assert that their theorem extends to n-dimensional dynamical systems, and their proof still holds with little modification.

Theorem 4. [Góra & Boyarsky, 1998] *Given a continuous dynamical system $f : M \to M$, $f \in C^0(M)$, and a sequence of convergent transformations, $f_n \to f$, in the C^0 topology, then if μ_n are each f_n-invariant, then a weak-* limit μ of $\{\mu_n\}$ is f-invariant.*

Proof. To show that μ is f-invariant, we must show that $\int_M h d\mu = \int_M h \circ f d\mu$ for the set of "test functions," all $h \in C^0(M)$ with compact support. We bound,

$$\begin{aligned} &\left| \int h d\mu - \int h \circ f d\mu \right| \\ &\quad \le \left| \int h d\mu - \int h d\mu_n \right| + \left| \int h d\mu_n - \int h \circ f_n d\mu_n \right| \\ &\qquad + \left| \int h \circ f_n d\mu_n - \int h \circ f d\mu_n \right| \\ &\qquad + \left| \int h \circ f d\mu_n - \int h \circ f d\mu \right| \end{aligned} \tag{22}$$

By assumption, μ_n is f_n-invariant, and so the second term is identically zero. The first and fourth terms converge to zero by definition of weak-* convergence of $\mu_n \to \mu$. By continuity of h, the third term may also be bounded by an arbitrarily small term proportional to the modulus of continuity of h: $|h \circ f_n(x) - h \circ f(x)| \le \omega(\delta) = \sup\{|h(y) - h(y')| : d(y, y') \le \delta\}$ where we choose $\delta = \|f - f_n\|_{C^0(M)}$. ■

Remark. Since a chaotic set is characterized by an infinite number of periodic orbits, f typically has infinitely many invariant atomic measures. In addition, the unstable chaotic saddle sets are the Cantor-like sets which arise by dynamically removing a subset $S \subset M$, and all of its preimages [Lai *et al.*, 1993] $M - \cup_{i=0}^{\infty} f^{-i}(S)$, when this is a nonempty uncountable set. The notation f^{-i} is taken to be the possibly multivalued preimage, if the inverse of f does not exist. In fact, these unstable chaotic saddles may support an infinite number of measures [Lai, 1997]. One of our goals in this paper is to choose one of these invariant measures, and stabilize it, i.e. we find a nearby dynamical system such that it has the desired measure as its "physical" measure.

Remark. The converse of the above theorem is that a bounded away from zero perturbation is required to control a noninvariant density: if μ is not f-invariant, then one cannot find an arbitrarily close g to f, such that μ is g-invariant. Given an arbitrary μ, there exists *some* $g : M \to M$ such that μ is g-invariant. This can be considered as a highly undetermined version of the IFPP, on a sequence of refining grids covering M, amounting to a subset of our control problem, using rules 1 and 2, but not 3 and 4. Thus large $\|f - g\|_{\sup} \sim O(1)$ perturbations are allowed, and all entries of the stochastic matrix on the grid may be altered. Obviously, there may be infinitely many matrix solutions; one simple solution can be explicitly constructed in terms of the so-called "3-band" transformations [Góra & Boyarsky, 1993, 1997]. Nonzero entries are placed on the three-band diagonal, corresponding to a digraph in which each node is connected to its left and right neighbors, and to itself. The construction given in [Góra & Boyarsky, 1993] chooses these arc weights to select any dominant eigenvector. Again, this eigenvector describes a discrete density ρ_n, and a sequence of these constructions formulate a sequence of $g_n \to g$ as $\varepsilon \to 0$, with μ_n which are g-invariant and μ_n converge weakly to μ.

In summary, if we are willing to perturb by a lot, then there is a g to be found, but for an arbitrary target μ, a large perturbation is required, if μ is not f-invariant.

7. Examples

We now give examples to demonstrate one-dimensional and then two-dimensional invariant measure stabilization.

Consider the *logistic map,* $x_{n+1} = 4x_n(1 - x_n)$, well known to have "fully developed chaos" for this parameter value [Devaney, 1989], $\lambda = 4$. Furthermore, this logistic map has the well-known unique absolutely continuous invariant density function $\rho(x) = 1/\pi\sqrt{x(1-x)}$, and therefore this is the unique, bounded from zero, fixed point of the Frobenius–Perron operator [Lasota & Mackey, 1997]. Using an evenly spaced grid of $n = 50$ cells, on the interval $I = [0, 1]$, and a chaotic test orbit $\{x_i\}_{i=0}^{10^5}$, we formed the associated stochastic transition matrix A with stationary eigenvector $\mathbf{v}$ shown in Fig. 4.

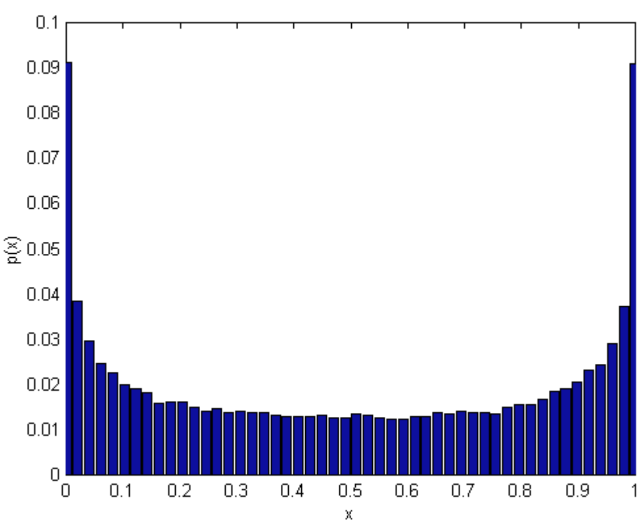

Fig. 4. Invariant density of the logistic map $x_{n+1} = 4x_n - x_n$, obtained by the fixed point of the approximate Frobenius–Perron operator, derived from a 10^5 iterate test orbit, and an equipartition $\cup_{i=0}^{50} Q_i = [0, 1]$. This sampled density is very close to the expected density $\rho(x) = 1/\pi\sqrt{x(1-x)}$.

From Remark of Theorem 4, we have that any periodic orbit supports an atomic invariant measure, and we propose to stabilize densities corresponding to arbitrarily chosen periodic orbits:

Example 1. (*Logistic map*) We choose to stabilize the cells which contain the two iterates of the period-2 orbit: $x = (5 - \sqrt{5})/8 \approx 0.345492$, $f(x) = (5 + \sqrt{5})/8 \approx 0.904508$. Therefore, we choose a cumulative target distribution vector, $(\mathbf{v} + \delta\mathbf{v})_{18} = (\mathbf{v} + \delta\mathbf{v})_{46} = 0.5$, and $(\mathbf{v} + \delta\mathbf{v})_i = 0$ if $i \neq 18$ or 46. Note that the sign of the derivative is unimportant to the FP operator, Eq. (2), and both controlling maps give the same invariant density. In Fig. 5 we show the stabilized density of the piecewise-linear function approximates $f_{50}(x)$ shown in Fig. 5 insets. By construction, the perturbation error is bounded $\sup_{x\in[0,1]} |f_{50}(x) - 4x(1 - x)| \leq 1/n = 1/50$ according to Eq. (19). Other periodic orbits are equally accessible.

We are not restricted to densities supporting periodic orbits. We now parameterize a family of target densities which "spread" density from one region, $(x > 0.5)$, to another, $(x \leq 0.5)$.

Example 2. (*Destroying Symmetry in the Logistic map*) We find that rather arbitrary, but small, variations $\delta\mathbf{v}$ give control feasible targets. Given $n = 20$ cells, let us choose $(\delta\mathbf{v})_i = \varepsilon$ if $i \leq 10$ and $(\delta\mathbf{v})_i = -\varepsilon$ otherwise. When $\varepsilon = 0$, the target is the original invariant measure, and no control is applied. For small target variations, $\varepsilon \leq \varepsilon_{\mathrm{cr}_{20}} = 0.0004$, we find that the control objective is feasible; Eq. (14) yields a stochastic matrix. See in Fig. 6 the successfully targeted distribution (dashed), and the corresponding piecewise-linear map, $f_{20}(x)$. We have changed the originally symmetric distribution of the logistic map, to one with 47.25% of the probability on the left of $x = 0.5$, and 52.75% on the right. See also that $\varepsilon \leq \varepsilon_{\mathrm{cr}_{10}} = 0.0061$ (solid) when $n = 10$ cells. In both of these extreme cases, the hyperplane was pushed to the edge of the stochastic box (see Fig. 1 and Theorem 2), and correspondingly we see that the control maps $f_{10}(x)$ and $f_{20}(x)$ were forced to have points with almost vertical derivatives.

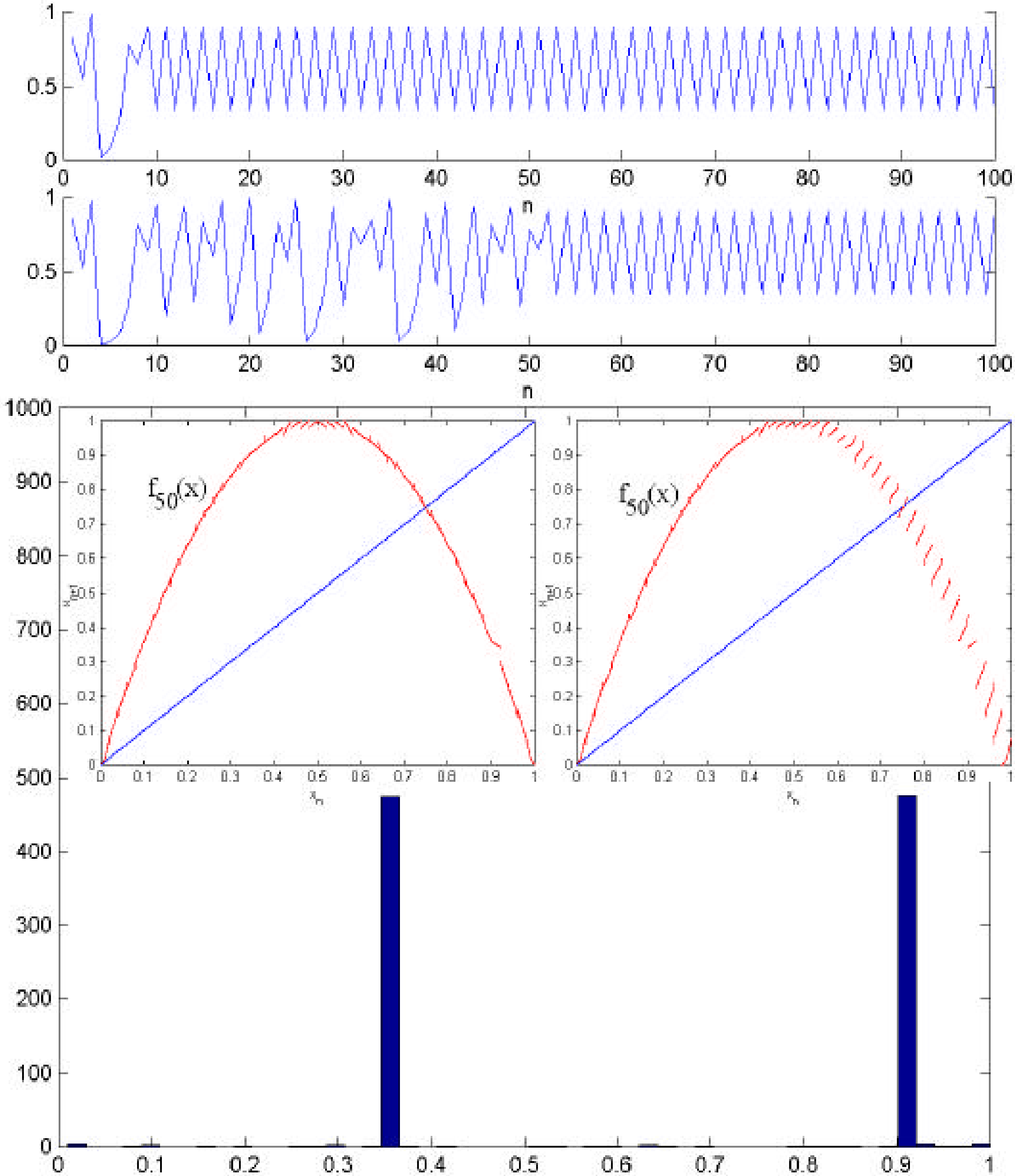

Fig. 5. Nearby the dynamical system $f(x) = 4x(1 - x)$, with invariant density shown in Fig. 4, we find a piecewise-linear map $f_{50}(x)$ with invariant density supported on the cells Q_{18} and Q_{46}, containing the period-2 orbit of the logistic map. Insets: Perturbed maps $f_{50}(x)$ with invariant density shown. Above: Time-series of trajectories under controlled map from two "typical" initial conditions.

We have achieved many other families of noninvariant target densities, including concentrating density only partly over a point. Always we find that noninvariant targets require some $\varepsilon_{\mathrm{cr}} > 0$, before our algorithm breaks-down as predicted by Remark of Theorem 4.

A main theme of this paper is that not all arbitrary target distributions are supported by an arbitrary transition matrix A. Another theme is that coarsening the grid (fewer cells) allows a wider latitude of target distributions. See Theorem 4.

Example 3. (*Cell Refinement and the Logistic map*) Given $n = 20$ cells, one cannot find a nearby dynamical system which approximates a uniform distribution: $(\mathbf{v})_i = 1/20$. Similarly, we cannot

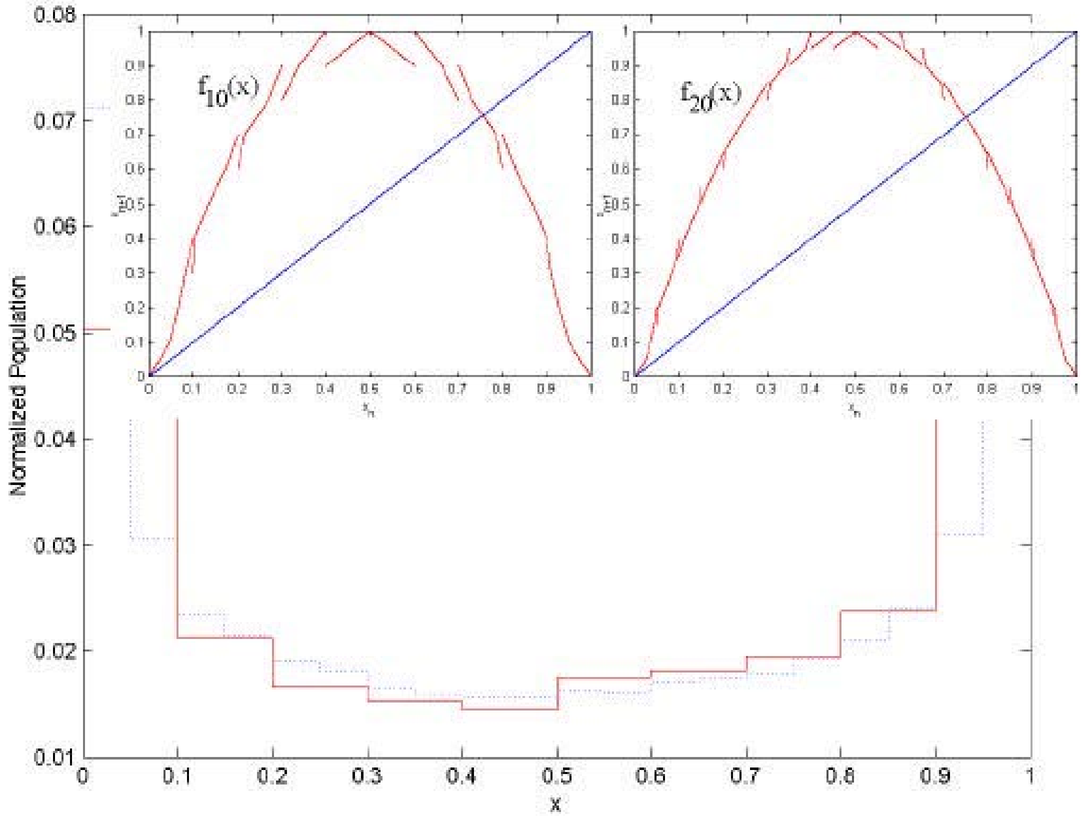

Fig. 6. Maximal distribution variations of Logistic map from asymmetric target families $(\delta\mathbf{v})_i = \varepsilon$ if $x \leq 0.5$ and $(\delta\mathbf{v})_i = -\varepsilon$ otherwise, where $\varepsilon_{\mathrm{cr}_{10}} = 0.0061$ (solid) and $\varepsilon_{\mathrm{cr}_{20}} = 0.0004$ (dashed) with $n = 10$ and 20 cells respectively. Compare to Fig. 5. Above left: Control map $f_{10}(x)$. Above right: Control map $f_{20}(x)$.

stabilize other famous, named targets. However, coarsening the grid eventually allows *all* targets. Just one cell, $n = 1$, obviously allows all targets for all maps in this trivial case, but the required perturbation to the map is then of the order of the phase space. For the logistic map, the grammar on two cells also allows all targets since the two by two transition matrix has all four entries nonzero. Somewhere between 20 and 2 cells, Example 2 can be controlled beyond $\varepsilon_{\mathrm{cr}} = 0.0004$, e.g. with $n = 10$ cells, we find critical value $\varepsilon_{\mathrm{cr}} = 0.0061$.

In the space of stochastic matrices, we interpret loss of controllibility of a noninvariant density target, with refining ε-grids, as follows. Referring to Fig. 1, we conjecture that an ε-parameterized family of target densities also ε-parameterizes a continuously moving hyperplane, $D_\varepsilon \cdot \boldsymbol{\delta}\mathbf{a} = \mathbf{z}$, that is pushed outside the box by increasing ε.

To consider two-dimensional measure targeting, we start with the well-known Hénon map [Henon, 1976]: $(x, y) \to f(x, y) = (1.4 - x^2 + 0.3y, x)$, which is widely believed to admit a chaotic attractor, but a rigorous proof of existence for an SBR-measure is still an open question. First we verify numerically that our piecewise-affine transformations f_n, constructed according to Eq. (21), do indeed have statistics which mimic the statistics of the original map.

Example 4. (*Comparison between Piecewise-Affine Models and Nonlinear Maps*) In Fig. 7, we give numerical evidence to support Conjecture 1. The piecewise-affine models have statistics converging to the observed statistics of the nonlinear models, using standard map and Henon map data sets, both generated on a 50×50 grid. We show the average error improvement (lines with circles) as a function of test orbit length, between the piecewise-affine models and the true nonlinear map. The average error is $(\mu(Q_i) - \mu_n(Q_i))/\mu(Q_i)$, where $\mu(Q_i)$ denotes the observed cell Q_i occupancy of the N iterate test orbit under the true nonlinear map f, and $\mu_n(Q_i)$ denotes the same thing for the model f_n. The lines with x's show the maximum worst $(\mu(Q_i) - \mu_n(Q_i))/\mu(Q_i)$. The decreasing

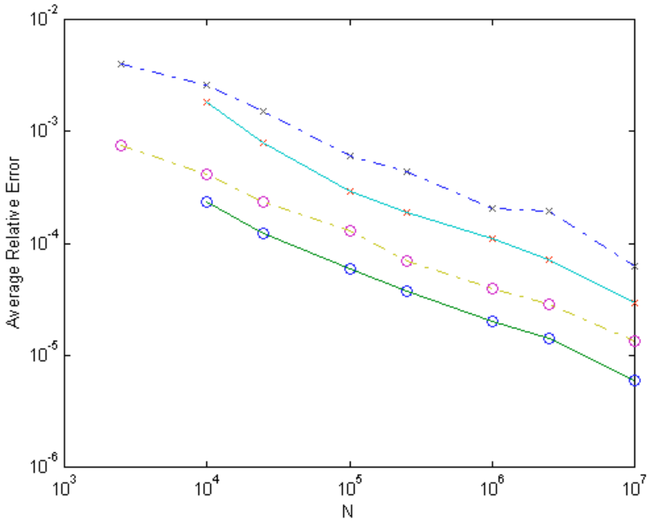

Fig. 7. Improving error statistics as a function of test orbit length, between the piecewise-affine models in Eq. (21) and the true nonlinear map. The solid lines show standard map data, and the dashed lines show Hénon map data, both generated on a 50×50 grid $\cup_i Q_i$ grid. Lines with circles show average error $(\mu(Q_i) - \mu_n(Q_i))/\mu(Q_i)$ indexed by i over the grid Q_i, and lines with x's show the maximum worst $(\mu(Q_i) - \mu_n(Q_i))/\mu(Q_i)$.

error with increasing N is strong evidence supporting Conjecture 1.

In Fig. 8, we show the Hénon map's discrete distribution for $N = 10^5$ and the reasonably fine 50×50 grid, making each square cell Q_i, $(\varepsilon_x, \varepsilon_y) = (0.0716, 0.0716)$. Using a $50 \times 50 = 2500$ element grid, we might expect that a 2500×2500 FP matrix A would be required, but by only box counting occupied nodes, we save a tremendous amount of memory and computation: there are only 322 occupied cells, and the 322×322 matrix A is spares with only 656 nonzero elements. To remind us of the corresponding density function, $\rho(x, y)$, we have used a "3-D" impulse-plot representation, illustrating each density value $(\mathbf{v})_i$ over the corresponding grid square Q_i.

Example 5. (*Atomic measures of the Hénon map*) We choose to stabilize a density which is concentrated over, say, a period-1 orbit (see Fig. 9). The control variation is bounded by the grid size: $\|f_n(x, y) - f(x, y)\|_\infty \leq \varepsilon_x = \varepsilon_y$. Other periodic orbits are equally controllable.

Example 6. ("*Arbitrary*" *measures near the Hénon map*) In analogy to Example 2, we "destroy symmetry" by targeting a sequence of successively more asymmetric (and noninvariant to the Hénon

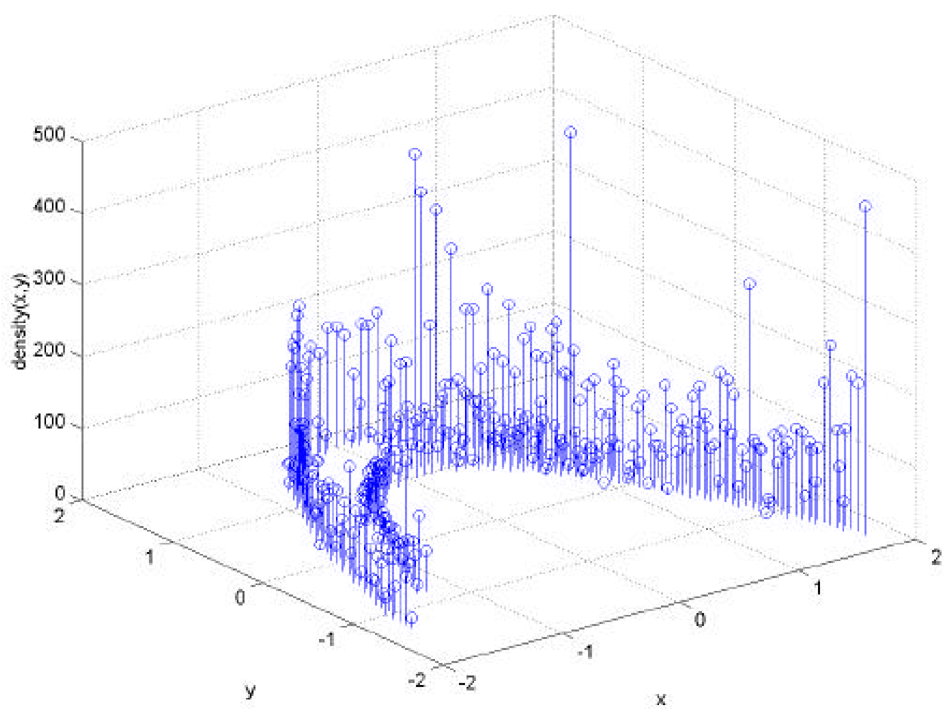

Fig. 8. A discrete distribution histogram of the Hénon map, using $N = 25,000$ and a 50×50 grid.

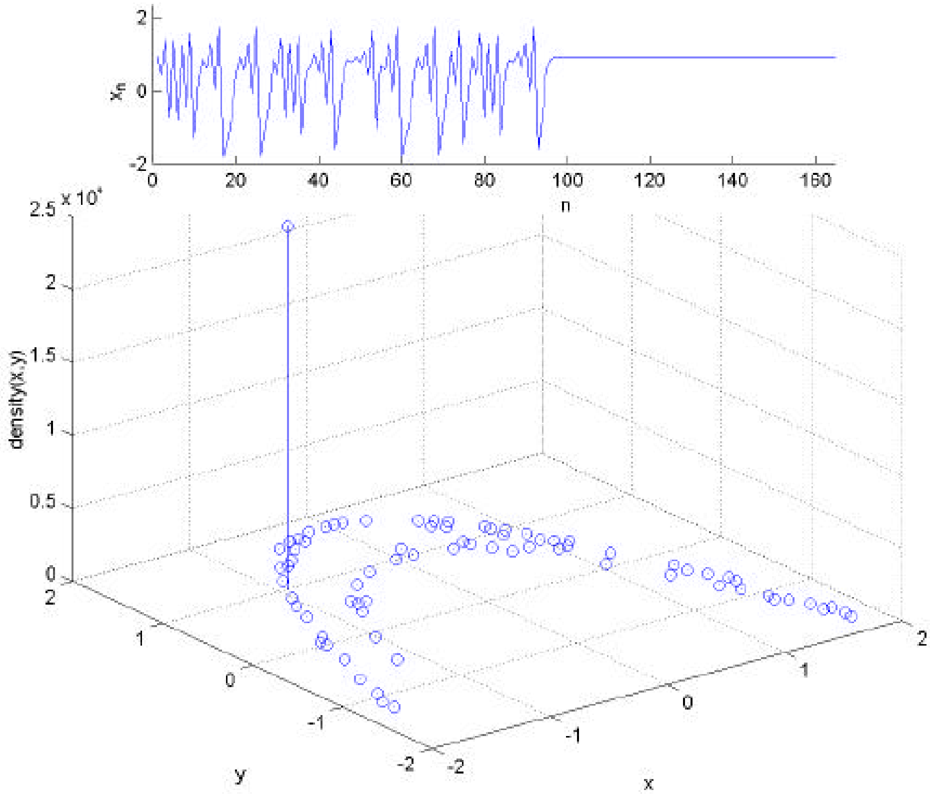

Fig. 9. Stabilized density histogram, $N = 25,000$, supported over the cell containing the period-1 orbit, controlled by a piecewise-affine map near the Hénon map, using a 50×50 grid. Above: An initial segment of a "typical" controlled x-time-series.

map), target densities. When attempting to shift mass evenly from the support of the attractor $y < 0$, to $y > 0$, we find a critical maximum amount of mass which may be "moved." For example, a 50×50 grid allows $\varepsilon_{\mathrm{cr}_{50}} = 4.3 \cdot 10^{-3}$ when the $y < 0$ measure is pushed evenly to $y > 0$, but the courser, 25×25 grid gave $\varepsilon_{\mathrm{cr}_{25}} = 7.4 \cdot 10^{-3}$, and the courser still 10×10 grid gave $\varepsilon_{\mathrm{cr}_{10}} = 1.1 \cdot 10^{-2}$.

Consider the well-known standard map [MacKay *et al.*, 1984], $(x, y) \to f(x, y) = ([y - (k/2\pi)\sin(2\pi kx) + x] \bmod 1,\ [y - (k/2\pi) \sin(2\pi kx)] \bmod 1$. The phase space of the standard map is the unit torus. We chose this example to emphasize the flexibility of our algorithm, because it is not known whether an absolutely continuous invariant measure exists.

Example 7. (*Standard map*) In Fig. 10(a), we show a discrete density of the Standard map, due to sampling a $25,000$ iterate orbit, on a 35×35 grid covering the unit torus. The resulting transition matrix is 831×831, with 2969 nonzero entries. Given the relatively short sample, and notoriously slow transport of area preserving maps due to "sticky" invariant tori and cantori [MacKay *et al.*, 1994], this is not expected to be even close to an invariant density, *if* one exists. The relatively higher density band through the middle of the picture, is due to the "stickiness" of the two golden mean resonances ($\omega = 1/\gamma$, and $1/(\gamma^2)$, where $\gamma = (1 + \sqrt{5})/2$) [MacKay *et al.*, 1984]. Figure 7 shows that the piecewise-affine models well approximate these densities. We select cell Q_{401}, the transition matrix has a nonzero element on the diagonal, $A_{401,401} > 0$, and hence includes a fixed point. Figure 10(b) shows a successfully stabilized density, by a piecewise-affine map nearby the standard map, $\|f - f_{831}\|_\infty \le \varepsilon_x = \varepsilon_y = 1/35$.

8. Invariant Set Stabilization

If we wish to target an invariant measure μ supported over an invariant set S, but we do not know what this measure is, then we cannot apply the algorithm developed in Sec. 3. An explicit target distribution would be required *a priori*. One might expect that choosing $\rho(x)$ distribution weights arbitrarily on the invariant set,

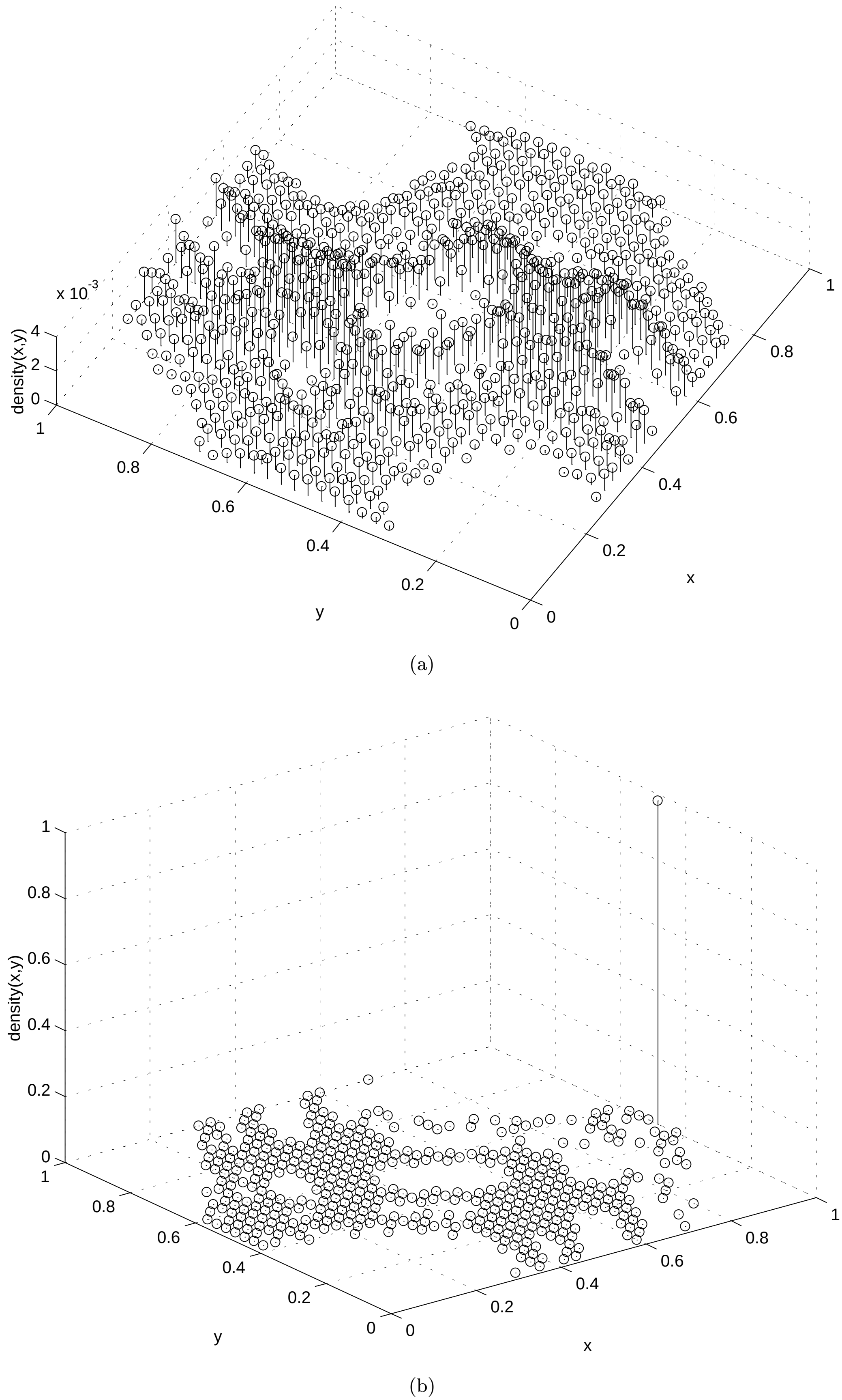

Fig. 10. (a) The discrete distribution for the Standard map, using $N = 2.5 \times 10^4$ and a 35×35 grid. (b) Stabilized density supported over the cell Q_{401} containing the period-1 orbit.

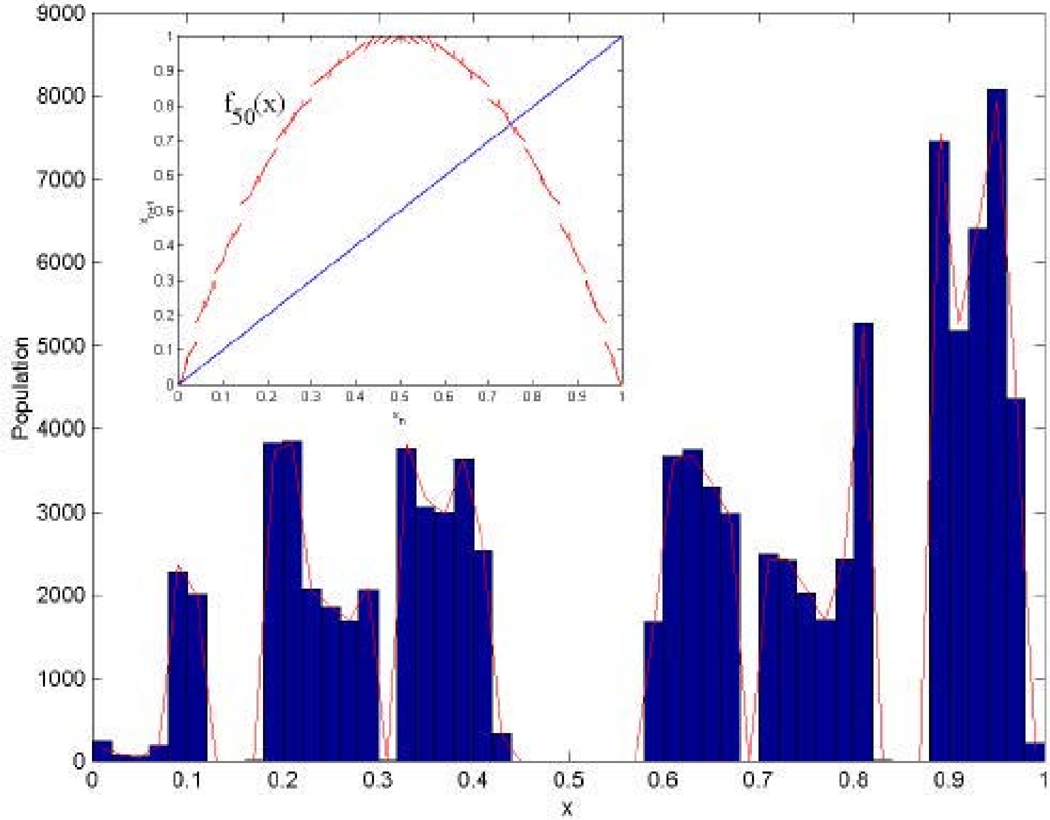

Fig. 11. Stabilized Chaotic Saddle in the Logistic Map. The stabilized distribution avoids the region $B = [0.44, 0.58]$, and pre-iterates. The histogram bar-plot of a 10^5 iterate controlled orbit agrees closely with predicted distribution curve from dominant eigenvector of $A_{\text{controlled}}$. Inset: The stabilized control map $f_{50}(x)$.

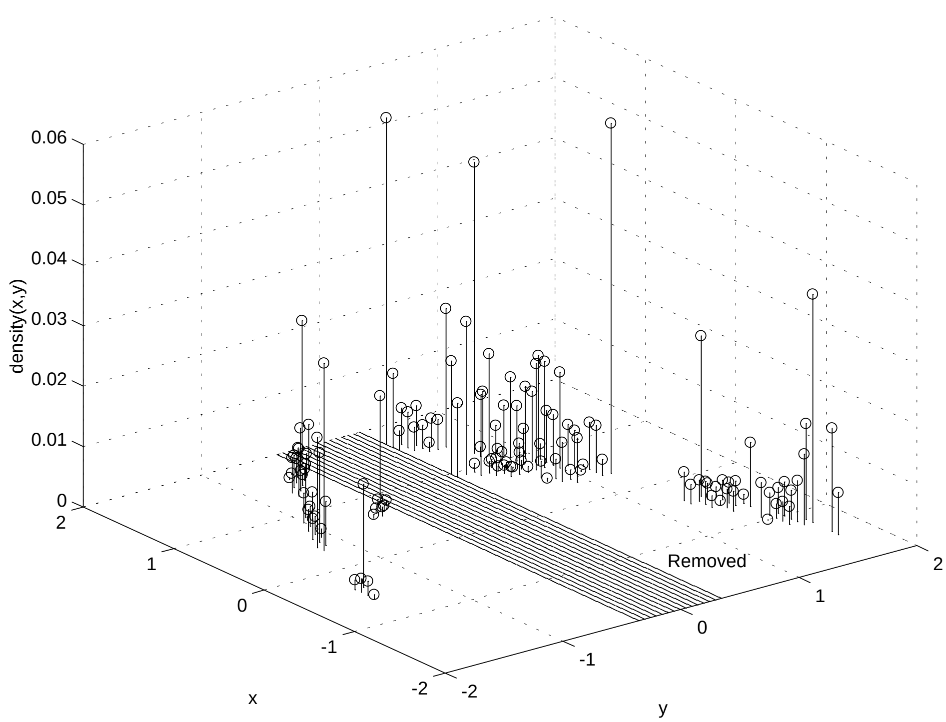

Fig. 12. Stabilized Chaotic Saddle in the Hénon Map. The stabilized distribution avoids the region $B = \{-0.358 \leq y \leq 0.358\}$, and preiterates/predecessors.

and $\rho(x) = 0$ if $x \notin S$, will allow application of our previous algorithm, but generally such an arbitrary distribution will not be f-invariant, and hence not attainable with an arbitrarily small perturbation. In this section, we give an alternative graph-based algorithm to target an invariant measure whose density values are allowed to "float."

We assume that we have already used Ulam's method to produce a stochastic matrix A. Suppose we wish to eliminate node Q_k from the equilibrium state. This can be achieved by eliminating all arcs into node Q_k by associating zero probability to all arcs into Q_k. This is effectively achieved by placing zero's in the kth row.

$$\text{Let} \quad A_{k,i} = 0, \quad \text{for all} \quad i\,. \tag{23}$$

To recover a stochastic matrix, it is necessary to renormalize the columns of the newly altered A. We include the following very useful but trivial proposition for completeness.

Proposition 1. *The equilibrium eigenvector of the stochastic matrix A modified according to Eq.* (*23*) *has an equilibrium eigenstate,* $A \cdot \mathbf{v} = \mathbf{v}$, *in which* $(\mathbf{v})_k = 0$.

Proof. The $\lambda = 1$ eigenvector $\mathbf{v}$ exists since we renormalize A to a stochastic matrix. The fact that $(\mathbf{v})_k = 0$ trivially follows Eq. (23) and the matrix row multiplication, $(\mathbf{v})_k \equiv \sum_i A_{k,i}(\mathbf{v})_i = 0$. ■

Remark. Proposition 1 can be multiply-applied to eliminate large sections of phase space, by eliminating multiple nodes/grid cells. However, an eliminated node may cause other nodes to be eliminated in a "domino-like" effect; the eliminated node might have been the only access route to some other nodes, etc. Hence, if too many nodes are eliminated, then the resulting invariant set tends to be empty.

What we are describing in directed graph language, is dynamic removal of set B in state space, and its preimages $f^{-i}(B)$. Often, these preimage sets will overlap, and if B is too large, then the above process will effectively remove all of the nodes, meaning there exists no transformation f_n to achieve the goal of avoiding B.

Remark. An invariant set S of f which avoids the set $B \in M$ can be written

$$S = M - \cup_{i=0}^{\infty} f^{-i}(B)\,. \tag{24}$$

Such sets are typically Cantor-like unstable chaotic saddles, if they are nonempty. Given a fixed grid $\cup_i Q_i$, of cell diameter ε, then the minimal cell cover of S can be found, $S \in \cup_k Q_{i_k}$. The associated Frobenius–Perron matrix can be found by Proposition 1, in which the equilibrium eigenstate has $(\mathbf{v})_{i_k} = 0$ for each k. Once the matrix A has been formed, a corresponding transformation on the grid is found by the IUP described in Sec. 5. By refining the grid $\varepsilon \to 0$, a sequence of transformations can be constructed whose invariant densities limit on an invariant density on the approximate of the Cantor set S.

Example 8. (*Stabilized Chaotic Saddles in the Logistic Map and the Hénon Map*) We remove the region $B = [0.44, 0.58]$, and pre-iterates in the logistic map, Fig. 11 vice Fig. 4, and we eliminate the region $B = \{-0.358 \leq y \leq 0.358\}$, and preiteraes/predecessors in the Hénon Map, Fig. 12 vice Fig. 8.

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