COMPARISON OF DIFFERENTIAL GEOMETRY PERSPECTIVE 
OF SHAPE COHERENCE BY NONHYPERBOLIC SPLITTING 
TO COHERENT PAIRS AND GEODESICS

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ABSTRACT. Mixing and coherence are fundamental issues at the heart of understanding fluid dynamics and other non-autonomous dynamical systems. Recently the notion of coherence becomes a more rigorous footing, in particular, within the studies of finite-time nonautonomous dynamical systems. Here we recall “shape coherent sets” proved correspond to slowly evolving curvature, for which tangency of finite time stable foliations (related to a “forward time” perspective) and finite time unstable foliations (related to a “backwards time” perspective) serve a central role. We developed zero-angle curves, meaning non-hyperbolic splitting, by continuation methods in terms of the implicit function theorem, from which follows a simple ODE description of the boundaries of shape coherent sets. We also compared and contrasted this perspective to both the variational method of geodesics [14], as well as the coherent pairs perspective [10] from transfer operators.

1. Introduction. Understanding and describing mixing and transport in two-dimensional fluid flows has been a classic problem in dynamical systems for decades. Here we focus on theories related to coherence in finite-time nonautonomous systems, particularly three recent theories which are the geodesic theory of transport [14] whereby initially described that transport barriers relate to minimally stretching material lines (“Seeking transport barriers as minimally stretching material lines, we obtain that such barriers must be shadowed by minimal geodesics under the Riemannian metric induced by the CauchyGreen strain tensor.” [14]) and later it was improved to “stationary values of the averaged strain and the averaged shear,” [15], coherent pairs [10, 18, 1] viewed by transfer operators in terms of evolving density, and shape coherent sets [19] where the nonlinear flow itself as considered restricted to special “shape coherent sets” reveal that the otherwise complicated

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flow reduces to a simpler transformation, namely rigid body transformations, on the corresponding time and spatial scales. See Tables 1.

We recently introduced a definition concerning coherence called shape coherent sets\(^1\), motivated by an intuitive idea of sets that "hold together" through finite-time. We connected shape coherent sets to slowly evolving boundary curvature by studying the tangency of finite-time stable and unstable foliations, which relate to forward and backward time perspectives respectively. The zero-angle curve which is developed from the nonhyperbolic splitting stable and unstable foliations by continuation methods that relate to the implicit function theorem. These closed zero-angle curves preserve their shapes in the time-dependent flow by relatively small change of curvatures. See \[19\], on calculating the shape coherent factor.

In this paper, we contrast these three perspectives of coherence by calculating several measures which are the evolution of arc length, the relative coherence pairing, foliation angles, change of curvature, registration of shapes and the shape coherent factor for sets developed from each of these methods, respectively, two of which are shown in Table 1-3. We show that each of these three perspectives of coherence may keep their advantages in their corresponding measures. By a geodesic perspective, arc length should vary slowly. On the other hand, by perspective of shape coherent sets, shape should be roughly preserved and the nonlinear flow restricted to that set should appear as a rigid body motion on that scale of time and space. This may allow arc length to vary, however generally slowly. For coherent pairs, the definition \[10\] allows perfect overlap of a set and its image, so an extra condition called "robust" must be added, which was later clarified in \[9, 11\] that effectively numerical diffusion was introduced in the stage of estimating the Ulam-Galerkin matrices effectively rewards sets whose boundary curves do not grow dramatically, thus an implicit connection between the geodesic theory and the coherent pairs theory. See also \[8\]. Table 2-3 summarize a contrast study for benchmark examples, the Rossby wave and the double gyre system, where for best possible comparison, a specific "coherent set" was found that seems to be roughly the same set as identified by all three perspectives. Therefore, the definition in each is not identical. Furthermore, there are numerical estimation issues that likely vary between each approach, as especially seen for example in the transfer operator method since many cells were required, and therefore many orbit samples for a reasonable estimate of a boundary curve.

It may seem striking that each method is performing well in the measures of the other methods. There are of course differences between the methods as well, since each tends to identify sets but the other two do not, with significant difference in the non-corresponding measure. Note that we have used LCSTool \[16\] that uses closed shearlines, closed null-geodesics of the Green-Lagrange strain tensor, \[15\] which was as pointed out to be for incompressible flows, these are infinitesimally arc length conserving, but not generally.

At last, in the appendix, we show a simple example to demonstrate that there exist different shapes with the same area and the same arc length.

2. Review of Shape Coherence, Relative Coherence and Geodesic Transport Barrier. Let \((\Omega, A, \mu)\) be a measure space, where \(A\) is a \(\sigma\)-algebra and \(\mu\) is a normalized measure that is not necessarily invariant. We assume that \(\Omega \subset \mathbb{R}^2\).

\(^1\)In this paper, we always use the initial status of a closed curve as its reference sets, i.e. as \(B\) in Eqn. (2).
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<table>
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Table 1

See [10, 18, 19]. Consider the planar nonautonomous differential equation [14],

\[ \dot{x} = v(x, t), \quad x \in U \subset \mathbb{R}^2, t \in [t_-, t_+], \]

We use \( \Phi \) represents the flow map of the system\(^2\).

2.1. Shape Coherence. Here we review shape coherence [19]. Taking a Lagrangian perspective, we offer the following mathematical definition:

**Definition 2.1.** [19] **Finite Time Shape Coherence** The shape coherence factor \( \alpha \) between two measurable nonempty sets \( A \) and \( B \) under a flow \( \Phi_t \) after a finite time epoch \( t \in 0 : T \) is,

\[ \alpha(A, B, T) := \sup_{S(B)} \frac{m(S(B) \cap \Phi_T(A))}{m(B)} \]

where \( S(B) \) is a group of transformations of rigid body motions of \( B \), specifically translations and rotations descriptive of *frame invariance*. We interpret \( m(\cdot) \) to denote Lebesgue measure, but one may substitute other measures as desired. Then we say \( A \) is finite time shape coherent to \( B \) with the factor \( \alpha \) under the flow \( \Phi_T \) after the time epoch \( T \), but we may say for short that \( A \) is shape coherent to \( B \). We shall call \( B \) to be the *reference set*, and \( A \) shall be called the *dynamic set*.

In other words, if the flow \( \Phi_T \) is restricted to a shape coherent set \( A \), then \( \Phi_T|_A \) can be considered to be approximately equivalent to an easier transformation which belongs to the group \( S \), rigid body transformations. A stronger condition that can keep the shape coherence constantly through the whole time epoch instead of only for terminal times is stated as follow,

\(^2\)The form of \( \Phi \) may vary, such as \( \Phi_{t_0}^t, \Phi_T \) and \( \Phi(z, t; \tau) \), because we keep the original forms from different theories for better understanding. However, they represent the same flow.
**Definition 2.2.** ([19]) **Finite Time Shape Coherence Throughout the Time Epoch**

The shape coherence factor throughout the time epoch between between two measurable nonempty sets \(A\) and \(B\) under a flow \(\Phi_t\) throughout a complete finite time epoch \([t_1, t_2]\) is defined as,

\[
\beta(A, B, [t_1, t_2]) := \max_{t \in [t_1, t_2]} \alpha(A, B, t).
\]  

Let \(B\) be \(A\) itself, in this paper, which means we try to capture those sets with minimal shape change under the flow. Next, we introduce the finite-time stable and unstable foliations which are used to develop the boundaries of the shape coherent sets.

**Definition 2.3.** ([19]) Let \(\Phi_t(z)\) be a time-dependent flow and

\[
V_F(u) = \{v_F^\epsilon | v_F := \frac{w_F}{\|w_F\|} & w_F := \Phi_{-t}(\Phi_t(z) + \epsilon \vec{u}) - z\}
\]

\[
V_B(u) = \{v_B^\epsilon | v_B := \frac{w_B}{\|w_B\|} & w_B := \Phi_t(\Phi_{-t}(z) + \epsilon \vec{u}) - z\}
\]

be two sets of vectors, where \(z\) is a given point, \(\vec{u}\) is a vector, \(0 < t < +\infty\) and a given arbitrary small \(\epsilon > 0\). The finite-time stable and unstable foliations at \(z\), \(f^s_t(z) \in V_F\) and \(f^u_t(z) \in V_B\), are defined as follow,

\[
f^s_t(z) := \lim_{\epsilon \to 0^+} \arg \max_{\|\vec{v}\| = 1, \vec{v}^\epsilon \in V_F(u)} \|v_F^\epsilon\| \quad (4)
\]

\[
f^u_t(z) := \lim_{\epsilon \to 0^+} \arg \max_{\|\vec{v}\| = 1, \vec{v}^\epsilon \in V_B(u)} \|v_B^\epsilon\| \quad (5)
\]

when the limits exist.

And the included angle between pair of stable and unstable foliations is defined as follow,

**Definition 2.4.** ([19]) The included angle of the finite-time stable and unstable foliations is defined as \(\theta(z, t) : \Omega \times \mathbb{R}^+ \to [-\pi/2, \pi/2]\)

\[
\theta(z, t) := \arccos \frac{\langle f^s_t(z), f^u_t(z) \rangle}{\|f^s_t(z)\| \|f^u_t(z)\|} \quad (6)
\]

We show a comprehensive discussion on how the non-hyperbolic splitting angle of foliations preserves curvature of a curve in [19], and in turn, preservation of curvature yields significant shape coherence as noticed above. In order to generate curves of the zero-splitting angle, we apply the implicit function theorem to induce a continuation theorem which guarantees that we can use ODE solves and root-finding methods to describe the curve.

**Theorem 2.5** (Continuation Theorem). The set of \(z = (x, y)\) with \(\theta(z, t) = 0\) are a set of \(C^1\) curves, which can be written as \(C^1\) functions such as \(y = g_1(x)\) or \(x = g_2(y)\) of a finite \(t\), which depends on \(\theta_y \neq 0\) or \(\theta_x \neq 0\). Furthermore, \(dy/dx = -\theta_x/\theta_y\), for a given initial condition \(z_0\) has a solution \(g_1\) or likewise \(dx/dy = -\theta_y/\theta_x\) respectively.

For more details on the algorithm, see [19].
2.2. The Geodesic Theory of LCS. Next, we review the geodesic theory of Lagrangian coherent structures. The idea is that these structures should correspond to “least stretching curves”. Consider material length minimizers in the geodesic theory of transport barriers, an evolving material line \( \gamma_t \) in the system, has length

\[
I(\gamma_t) = \int_{\gamma_t} |dx| = \int_a^b |D\Phi_t(x)|dx
\]

\[
= \int_{s_1}^{s_2} \sqrt{(r', C_t^{t_0}(r)r')ds}
\]

where \( D\Phi_t(x_0) \) denotes the derivative of the flow map, and

\[
C_t^{t_0} = (D\Phi_t(x_0))^T D\Phi_t(x_0)
\]

denotes the Cauchy-Green strain tensor, with \( T \) referring to the matrix transpose.

In \cite{14}, a transport barrier of system \( 1 \) over the time interval \([t_0, t]\) is a material line \( \gamma_t \), whose initial position \( \gamma_0 \) is a minimizer of the length functional \( I_{t_0} \) under the boundary conditions,

\[
r(s_2) \neq r(s_1), \quad r'(s_i) = \lambda r(s_i), \quad i = 1, 2
\]

\[
h(s_{1,2}) = 0, \quad r'(s_1) = \eta_\pm(r(s_1)), \quad r'(s_2) = \eta_\pm(r(s_2)), \quad r(s_1) \neq r(s_2),
\]

where \( \lambda_{1,2} \) are the eigenvectors corresponding to the smaller and larger eigenvalues \( \lambda_1, \lambda_2 \) of \( C_t^{t_0}(r(s_{1,2})) \), \( h(s_{1,2}) \) is a pointwise normal, smooth perturbation to \( \gamma_0 \) and \( \eta_\pm \) are the normalized Lagrangian shear vector fields, which are defined as,

\[
\eta_\pm = \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} \xi_1 \pm \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} \xi_2.
\]

And a transport barrier is a hyperbolic barrier if \( \gamma_0 \) satisfies the hyperbolic boundary conditions as defined in Eqn. \( 9 \). A transport barrier is a shear barrier if \( \gamma_0 \) satisfies the shear boundary conditions as defined in Eqn. \( 10 \).

For the computational part, for comparison here we use the LCS tools which is developed by the Nonlinear Dynamical Systems Group at ETH Zurich, led by Prof. George Haller. See. \cite{16}.

2.3. The Relatively Coherent Pairs. In this section, we briefly review the relatively coherent pairs. Relatively coherent pairs \cite{18} describe a system by hierarchical partitions based on the idea of coherent pairs. See \cite{10}. Given the time-dependent flow \( \Phi(z, t, \tau) : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \Omega \), through the time epoch \( \tau \) of an initial point \( z \) at time \( t \), a coherent pairs \( (A_t, A_{t+\tau}) \) can be considered as a pair of subsets of \( \Omega \) such that,

\[
\Phi(A_t, t, \tau) \approx A_{t+\tau}.
\]

**Definition 2.6.** \cite{10} \( (A_t, A_{t+\tau}) \) is a \( (\rho_0, t, \tau) \)-coherent pair if

\[
\rho_{\mu}(A_t, A_{t+\tau}) := \mu(A_t \cap \Phi(A_{t+\tau}, t + \tau, -\tau))/\mu(A_t) \geq \rho_0
\]

where the pair \( (A_t, A_{t+\tau}) \) are ‘robust’ to small perturbation and \( \mu(A_t) = \mu(A_{t+\tau}) \).
Then we build a relative measure on $K$ induced by $\mu$, where $K$ is a nonempty measurable subset of $\Omega$. In this way enters refinements of the initial partition on successive scales. A relative measure of $K$ to $\Omega$ is, \[ \mu_K(A) := \frac{\mu(A \cap K)}{\mu(K)} \] for all $A \in \mathcal{A}$. From the above definition, it follows that $(K, \mathcal{A}|_K, \mu_K)$ is also a measure space, where $\mathcal{A}|_K$ is the restriction of $M$ to $\mathcal{A}$ and $\mu_K$ is a normalized measure on $K$. We call the space $(K, \mathcal{A}|_K, \mu_K)$, the relative measure space. Now, we define the relatively coherent pairs.

**Definition 2.7.** [18] Relatively coherent structures are those $(\rho_0, t, \tau)$-coherent pairs defined Definition 2.1, with respect to given relative measures on a subset $K \subset \Omega$, of a given scale, specializing [10].

To find relatively coherent structures in time-dependent dynamical systems, we use Frobenius-Perron operator. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\mu$ is a normalized Lebesgue measure. If $S: \Omega \to \Omega$ is a nonsingular transformation such that $\mu(S^{-1}(A)) = 0$ for all $A \in \mathcal{A}$ satisfying $\mu(A) = 0$, the unique operator $P: L^1(\Omega) \to L^1(\Omega)$ defined by, \[
\int_A Pf(x)\mu(dx) = \int_{S^{-1}(A)} f(x)\mu(dx)
\] for all $A \in \mathcal{A}$ is called the Frobenius-Perron operator corresponding to $S$, where $f(x) \in L^1(\Omega)$. See [17]. In our case, $S$ can be considered as the flow map $\Phi$ and the formula above can be written as \[
P_{t, \tau} f(z) := f(S^{-1}(z)) \cdot |\det D(S^{-1}(z))| = f(\Phi(z, t + \tau; -\tau)) \cdot |\det D\Phi(z, t + \tau; -\tau)|.
\] (15)

Suppose $X$ is a subset of $M$, let $Y$ be a set that includes $S(X)$. We develop partitions for $X$ and $Y$ respectively. In other words, let $\{B_i\}_{i=1}^m$ be a partition for $X$ and $\{C_j\}_{j=1}^n$ be a partition for $Y$. The Ulam-Galerkin matrix follows a well-known finite-rank approximation of the Frobenius-Perron operator, the entry of which is of the form \[
\hat{P}_{i,j} = \frac{\mu(B_i \cap S^{-1}(C_j))}{\mu(B_i)}
\] where $\mu$ is the normalized Lebesgue measure on $\Omega$. As usual, we numerically approximate $\hat{P}_{i,j}$ by, \[
P_{i,j} = \frac{\#\{x_k : x_k \in B_i \land S(x_k) \in C_j\}}{\#\{x_k : x_k \in B_i\}}
\] where the sequence $\{x_k\}$ is a set of test points (passive tracers). See [3].

The computation of relatively coherent sets is more complicated, since the boundary of these sets are small triangles. See [18]. A effective way to extract the boundary is to shrink the size of triangles; then we approximate the boundary by using the image processing toolbox and curve fitting toolbox in Matlab or connecting the center of triangles on the boundary.
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### Theory

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<th>Theory</th>
<th>Shape Coherent Factor</th>
<th>Arc Length Change (%)</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shape coherent sets</td>
<td>0.9637</td>
<td>1.14 %</td>
<td>The grid size is 200 × 200. See Fig. 1 left column.</td>
</tr>
<tr>
<td>Geodesic transport barrier</td>
<td>0.9362</td>
<td>0.93 %</td>
<td>Calculated by LCStool [16]. See Fig. 1 right column.</td>
</tr>
</tbody>
</table>
| Relatively coherent structures | 0.9210              | 1.9 %                 | 500000 by 500000 Ulam-Galerkin matrix with 2 × 10⁷ sample points. See Fig. 2.

**Table 2**

3. **Examples.** We next apply the three methods to the Rossby wave system and double gyre system, both of which are classic examples for studying and contrasting coherence and transport, [14, 10, 8, 18, 1].

3.1. **The Nonautonomous Double Gyre.** We consider the nonautonomous Double Gyre system,

\[
\begin{align*}
\dot{x} &= -\pi A \sin(\pi f(x,t)) \cos(\pi y) \\
\dot{y} &= \pi A \cos(\pi f(x,t)) \sin(\pi y) \frac{df}{dx}
\end{align*}
\]

where \( f(x,t) = \epsilon \sin(\omega t)x^2 + (1 - 2\epsilon \sin(\omega t))x, \epsilon = 0.1, A = 0.1 \) and \( \omega = 2\pi/10 \). See [21, 8]. Let the time interval be \([0, 20]\).

Table 2, Fig. 1 and Fig. 2 show the numerical results of the comparison among the three theories of the nonautonomous Double Gyre system. We can observe that the LCS keeps the arc length best of all, but the shape coherent sets has the highest shape coherent factor.

3.2. **An Idealized Stratospheric Flow.** The second benchmark we choose is a quasiperiodic system which represents an idealized zonal stratospheric flow [20, 10]. Consider the following Hamiltonian system

\[
\begin{align*}
\frac{dx}{dt} &= -\frac{\partial \Phi}{\partial y} \\
\frac{dy}{dt} &= \frac{\partial \Phi}{\partial x}
\end{align*}
\]

where

\[
\Phi(x, y, t) = c_3 y - U_0 \text{tanh}(y/L) + A_3 U_0 \text{sech}^2(y/L) \cos(k_1 x) + A_2 U_0 \text{sech}^2(y/L) \cos(k_2 x - \sigma_2 t) + A_1 U_0 \text{sech}^2(y/L) \cos(k_1 x - \sigma_1 t)
\]

Let \( U_0 = 41.31, c_2 = 0.205U_0, c_3 = 0.461U_0, A_3 = 0.3, A_2 = 0.1, A_1 = 0.075 \) and the other parameters be the same as stated in [20].

The numerical results in Table 3, Fig. 3 and Fig. 4 show that the elliptic LCS hold its arc length better than the other three curves also in this system, but the relatively coherent sets has the shape coherent factor a little better than the first shape coherent sets. However, comparing to the small region shape coherence, we
Figure 1. (A) and (B) show the initial and final status of a zero-splitting curve (Left) and an elliptic LCS (Right) around the zero-splitting curve. (C) and (D) are the comparison of foliation angles versus arc lengths between the two curves. The foliation angles of the zero-splitting curves are very small, but the angles of the geodesic curve vary. (E) and (F) are the curvature versus arc lengths. (G) and (H) are registrations of the curves. The arc lengths change of the zero-splitting curve and the elliptic LCS are 1.14% and 0.93%; and the shape coherent factors are 0.9637 and 0.9362.
Figure 2. (A) and (C) are the hierarchical partitions of the double gyre. The green curves around the upper left brown sets are the boundaries of our target relatively coherent sets. (B) and (D) are the foliation angles and change of curvature plot in different time. (E) shows the registration of the two curves. The arc lengths change is 1.9% and the shape coherent factor is 0.9210.

are more interested in shape coherence of bigger regions, so we show another much bigger shape coherent sets with the highest shape coherent factor, the second shape coherent sets. See Table 3.
<table>
<thead>
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<th>Arc Length Change (%)</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Shape coherent sets</td>
<td>0.9184</td>
<td>3.69 %</td>
<td>The grid size is $2000 \times 200$. See Fig. 3 left column.</td>
</tr>
<tr>
<td>2nd Shape coherent sets</td>
<td>0.9525</td>
<td>2.44 %</td>
<td>The grid size is $2000 \times 200$. See Fig. 4 left column.</td>
</tr>
<tr>
<td>Geodesic transport barrier</td>
<td>0.9156</td>
<td>1.2159 %</td>
<td>Calculated by LCS tool [16]. See Fig. 3 right column.</td>
</tr>
<tr>
<td>Relatively coherent structures</td>
<td>0.9193</td>
<td>1.2174 %</td>
<td>$80000 \times 80000$ Ulam-Galerkin matrix with $2 \times 10^7$ sample points. See Fig. 4 right column.</td>
</tr>
</tbody>
</table>

Table 3

4. Conclusions. In this paper, we have reviewed three complementary theories of coherence that come from different perspectives. These are the geodesic theory of the LCS which focus on the minimizer of material lines, the coherent pairs which describes “very small leakage” of sets and shape coherent sets which emphasize those sets preserve their shape by a slow evolving curvature. We then presented a comparing result of two benchmark examples, the double gyre and the Rossby wave. In the examples described here, it has been illustrated that all the three methods have reasonable and similar numerical results which agree with their own theories. For the arc length, the LCS always has the least change; and with respect to shape coherence, the zero-splitting curve has the best shape coherent number; the relatively coherent sets has the results very closed to them. Notice that here we only compared the elliptic shape for all the three methods, to allow each the best possibility of doing well relative to each other, but more investigations of more complicate sets may yield more divergent results.

5. Appendix. Here, we present an example to illustrate that two 2-D simply connected sets with same perimeter and area may have different shapes. By this statement, we argue that a closed geodesic transport barrier may lose its shape through the flow.

Fig. 5 shows two simply connect sets, a rectangle $A$ with height $a$ and width $a$ and a annulus sector $B$ with angle $\theta$, inner radius $r$ and outer radius $r + h$. The perimeter and area of $A$ are, $P_A$ and $S_A$,

\[
P_A = 2a + 2h \\
S_A = ah.
\]  

(21)

And the perimeter and area of $B$ are, $P_B$ and $S_B$,

\[
P_B = \frac{2\pi \theta (2r + h)}{360} + 2h \\
S_B = \frac{\pi \theta (2rh + h^2)}{360}.
\]  

(22)
Figure 3. (C) shows that the angle of zero-splitting curve keeps small. From (E) and (F) we can see that the zero-splitting curve has the less change of curvature. The zero-splitting curve is the closest one to the elliptic LCS, but there are still differences on size. Fig.4 shows a bigger zero-splitting curve. The arc lengths change of the zero-splitting curve and the elliptic LCS are 3.69\% and 1.2159\%; and the shape coherent factors are 0.9184 and 0.9156.

Now we assume the two shape A and B have the same area and perimeter, so we have the relationships as follow,

\[ P_A = P_B \quad \text{and} \quad S_A = S_B \iff a = \frac{\pi \theta (2r + h)}{360} \]  

(23)
We observe that the zero-splitting curve keep better curvature than the relatively coherent sets, though the shape of zero-splitting curve is bigger. The arc lengths change of the zero-splitting curve and the relatively coherent sets are 2.44\% and 1.2174\%; and the shape coherent factors are 0.9525 and 0.9193.

So, for example, let $h = 1$, $a = 3/2$, $r = 1$ and $\theta = 180/\pi$, we have $P_A = P_B = 5$ and $S_A = S_B = 3/2$. 
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