PAMM · Proc. Appl. Math. Mech. **11**, 855 – 856 (2011) / **DOI** 10.1002/pamm.201110415

A Stream Function Approach to Optical Flow with Applications to Fluid Transport Dynamics

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Optical flow is one of the classical problems in computer vision, but it has recently also been adapted to applications from other fields, such as fluid mechanics and dynamical systems. If the goal is to analyze the dynamics of system whose evolution is governed by a flow field that is the gradient of a potential function – which describes many flows in fluid dynamics – it is natural to approach the optical flow problem by reconstructing the potential function, also called the *stream function*, rather than reconstructing the components of the flow directly. This alternate approach allows one to impose scientific priors, via regularization, directly on the flow itself rather than on its components independently. We demonstrate the stream function formulation of optical flow and its application to reconstructing an oceanic fluid flow driven by satellite measurements. It is also shown how these flow fields can be used to analyze mixing and mass transport in the fluid system being imaged.

Stream Function Optical Flow

Optical flow is an image-based method for computing the apparent flow field governing a system evolution. Recently much work has been done using optical flow to analyze fluids, which requires a different physical model from the classical Horn-Schunk method [12]. Corpetti et al. [6, 7] proposed an alternative based on the continuity equation, resulting in an optical flow energy of the form

$$E(u, v) = \int_{\Omega} \left(I_t + \operatorname{div}\left(I\langle u, v\rangle\right)\right)^2 \, d\Omega + \alpha R(u, v), \tag{1}$$

where Ω is the spatial domain, $R(u, v)$ is an appropriate regularization scheme, and $\alpha > 0$ is the regularization parameter. This formulation has become common for analyzing fluid flows, for example in connection with and in comparison to particle image velocimetry (PIV) [5, 13], geophysical fluid flows [1, 2], and atmospheric motion [4]. While it is possible to incorporate the full Navier-Stokes equations for fluid flows in order to more accurately model the evolution of a fluid system [8], the model (1) is robust and results in more straightforward computational schemes.

Under the assumption that the flow being imaged is a "potential flow," i.e. that the flow field is the gradient of a potential, or *stream function*, it is natural to instead formulate the optical flow problem in terms of reconstructing the stream function directly. Within a Hamiltonian framework, the evolution equation governing the flow is thus $I_t = -\operatorname{div}\left(I \nabla_H \psi\right)$, where $\nabla_H \psi = \langle -\psi_y, \psi_x \rangle$ is the symplectic gradient. The resulting optical flow energy is

$$E(\psi) = \int_{\Omega} \left(I_t + \operatorname{div}\left(I \nabla_H \psi\right)\right)^2 \, d\Omega + \alpha R(\psi), \tag{2}$$

where $R(\psi)$ is a regularization of the stream function.

There are two primary goals of regularizing a variational optimization problem. The first goal is to ensure that the optimization problem is well-posed, i.e. that a solution exists, is unique, and is stable with respect to perturbations in the input data. For optical flow problems, this need not happen in general. In particular, for nearly any physically appropriate regularization, there exist data for which the governing flow field is not unique. The second reason for regularization – and one of the main advantages to computing optical flow via (2) – is to impose scientific priors directly on the flow. This is in stark contrast to (1), which enforces priors on the *components* of the flow. To see the distinction between these two formulations, suppose that the *a priori* assumption to be imposed is that the flow is sparse. The natural approach for the energy in (1) would be to regularize using the L^1 norm of u and v. This approach, however, favors a solution whose components, u and v, are sparse, but this need not result in a sparse flow. In particular, the complements of the supports of u and v need not overlap, which is to say that u may be 0 in many places and v may be 0 in many places, but they need not be 0 in the same places. Within the stream function formulation (2), it is straightforward to impose sparsity on the flow by regularizing via the total variation of the stream function.

Analyzing the Dynamics and Transport in an Oceanic System

Sea surface temperature can be simulated using a PDE ocean flow model with real data assimilation [15], and we compute the surface temperature "flow" using (2) with the classical smoothness regularization of Horn and Schunck. Two time instances from a single day in August 2002 and the corresponding flow field can be seen in Fig. 1 (a)-(c). The stream function formulation of the optical flow problem is especially suited to capturing the vortices in the flow.

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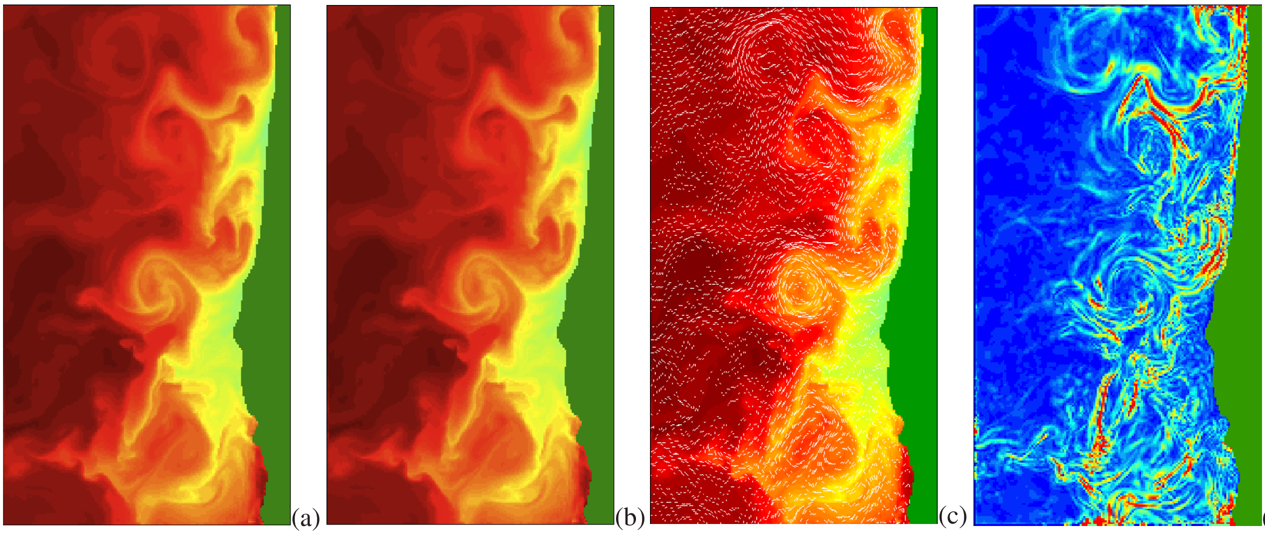

(a) (b) (c) (d)

Fig. 1 Images (a) and (b) show two time instances of sea surface temperature along the coast of Oregon (USA) from two different times from a single day in August 2002. Image (c) shows the optical flow field, computed using (2) with the classical smoothness regularization and regularization parameter $\alpha = 10^{-3}$, and image (d) shows the corresponding FTLE field over a 15-hour epoch. The red ridges in the FTLE field are pseudo-barriers, across which waters of different temperatures do not mix.

One of the primary applications of studying fluid flow is understanding mixing and mass transport. *Lagrangian coherent structures* (LCS) are artifacts in a flow that persist in time and impact the behavior of the fluid. The finite time Lyapunov exponent field is one example of an LCS; it's a scalar field whose high-value regions represent pseudo-barriers to mixing and mass transport. The details of LCS and FTLE can be found in [9–11, 14], and for a more detailed example of their use in analyzing oceanic mass transport, see [3]. In order to compute the FTLE field, a vector field describing the flow and a fixed time epoch over which to compute the the FTLE are required. The vector field is usually generated from a PDE model, so the ability to compute such a flow field directly from measured data allows to use FTLE's in a broader setting. The FTLE field for the sea surface temperature data, which was generated using only the computed optical flow field and not the PDE model, can be seen in Fig. 1 (d). (Videos of the time-varying optical flow field and the corresponding FTLE field can be found at http://nonlinear.cslabs.clarkson.edu/IBD/Results.htm.) The red ridges in Fig. 1 (d) are the high FTLE regions, representing pseudo-barriers to transport, transverse to which there is little flow. Essentially these are walls which (almost) stop mixing across the temperature gradient, but the walls open and close in time, allowing mixing through temporary corridor-like regions.

Acknowledgements The authors would like to thank S. Keeling and B. Forster for their efforts in organizing the GAMM 2011 symposium on mathematical image processing. The authors would also like to thank N. Tufillaro and the College of Oceanic and Atmospheric Sciences at Oregon State University for providing the sea surface temperature data. This research was supported by the United States Office of Naval Research under grant #N00014-09-1-0647.

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