Markov Parititions

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May 28, 2002

1 Why a Markov Partition?

To simplify analysis of a dynamical system, we often study a topologically equivalent system using symbol dynamics, representing trajectories by infinite length sequences using a finite number of symbols. (An example of this idea is that we often write real numbers as sequences of *digits*, a finite collection of symbols.) To represent the state space of a dynamical system with a finite number of symbols, we must *partition* the space into a finite number of elements and assign a symbol to each one. In probability theory, the term "Markov" denotes "memoryless." In other words the probability of each outcome conditioned on all previous history is equal to conditioning on only the current state; no previous history is necessary. The same idea has been adapted to dynamical systems theory to denote a partitioning of the state space so that all of the past information in the symbol sequence is contained in the current symbol, giving rise to the idea of a *Markov transformation*.

2 One-Dimensional Transformations

In the special, but important case that a transformation of the interval is Markov, the symbol dynamic is simply presented as a finite directed graph. A Markov transformation in $\mathcal{R}^1$ is defined as follows: [2].

Definition: Let $I = [c, d]$ and let $\tau : I \to I$. Let $\mathcal{P}$ be a partition of I given by the points $c = c_0 < c_1 < \ldots < c_p = d$. For $i = 1, \ldots, p$, let $I_i = (c_{i-1}, c_i)$ and denote the restriction of τ to I_i by τ_i. If τ_i is a homeomorphism from I_i onto a union of intervals of $\mathcal{P}$, then τ is said to be *Markov*. The partition $\mathcal{P}$ is said to be a *Markov partition* with respect to the function τ.

2.1 One-Dimensional Example

Map 1, (Fig 1a) is a Markov map with the associated partition $\{I_1, I_2, I_3, I_4\}$. The symbol dynamics are captured by the transition graph (Fig 1b). Although Map 2 (Fig 1c) is piecewise linear and is logically partitioned by the same intervals as map 1, the partition is not Markov because interval I_2 does not map onto (in the mathematical sense) a union of any of the intervals of the partition. However, we are not able to say that the map 2 is not Markov. There may be some other partition that satisfies the Markov condition. In general, finding a Markov partition or proving that such a partition does not exist, is a difficult problem.

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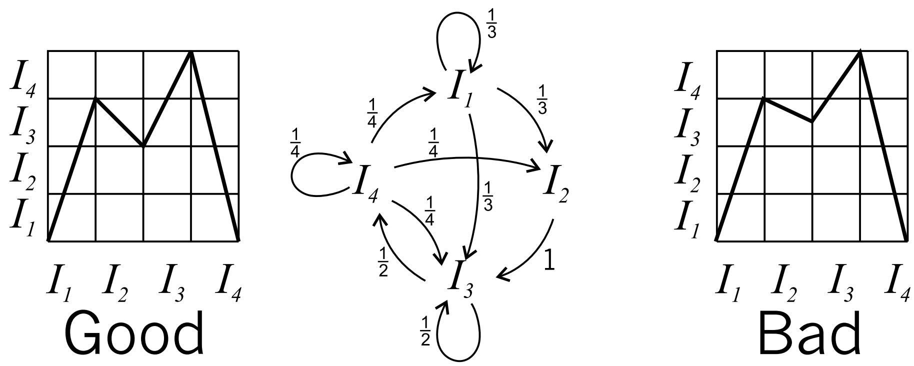

Figure 1: (a) A Markov map with partition shown. (b) The transition graph for map 1. (c) The partition is not Markov, because the image of I_2 is not equal to a union of intervals of the partition.

3 In Higher Dimensions

Definition A *Topological Partition* of a metric space M is a finite collection $\mathcal{P} = \{P_1, P_2, ..., P_r\}$ of disjoint open sets whose closures cover M in the sense that $M = \overline{P_1} \cup ... \cup \overline{P_r}$. [3]

Any topological partitioning of the state space will create a symbol dynamics for the map. In the special case where the partition is *Markov*, the symbol dynamics capture the essential dynamics of the original system.

Definition Given a metric space M and a map $f : M \to M$, a *Markov Partition* of M is topological partition of M into rectangles $\{R_1, ..., R_m\}$ such that whenever $x \in R_i$ and $f(x) \in R_j$, then $f[W^u(x) \cap R_i] \supset W_u[f(x)] \cap R_j)$ and $f[W^s(x) \cap R_i] \subset W_s[f(x)] \cap R_j$. [1, 7].

In simplified terms, this definition says that whenever an image rectangle intersects a partition element, the image must stretch completely across that element in the expanding directions and must be inside that partition element in the contracting direction. (See Fig 2.)

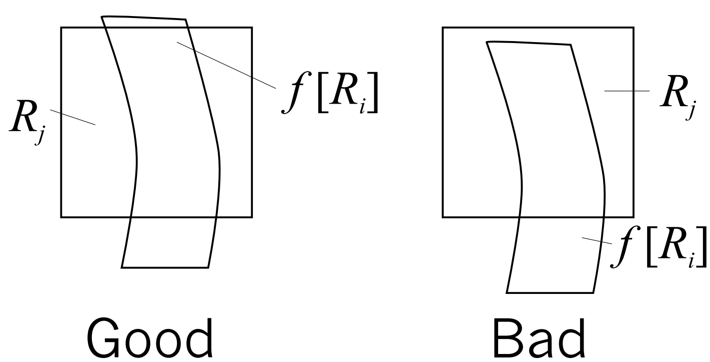

Figure 2: In the unstable (expanding) direction, the image rectangle must stretch completely across any of the partition rectangles that it intersects.

It is important to use a "good" partition so that the resulting symbolic dynamics of orbits through the partition well represents the dynamical system. If the partition is Markov, then "goodness" is most easily ensured. However, a broader notion, called *generating partition*, may be necessary to capture the dynamics. A *Markov partition* is *generating*, but the converse is not generally true. See [5, 6] for a thorough discussion of the role of partitions in representing dynamical systems.

3.1 Two-Dimensional Example - Toral Automorphism

The Cat Map, defined by

$$\mathsf{x} = (\mathsf{A}\mathsf{x}) \bmod 1 \tag{1}$$

$$\text{where } \mathsf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \tag{2}$$

yields a map from the unit square onto itself. This map is said to be on the toral space T^2 because the mod 1 operation causes the coordinate $1+z$ to be equivalent to z. A *Markov partition* for this map is shown in Fig 3. The *Cat Map* is part of a larger class of functions called toral Anosov diffeomorphisms, and [4] provides a detailed description of how to construct *Markov partitions* for this class of maps.

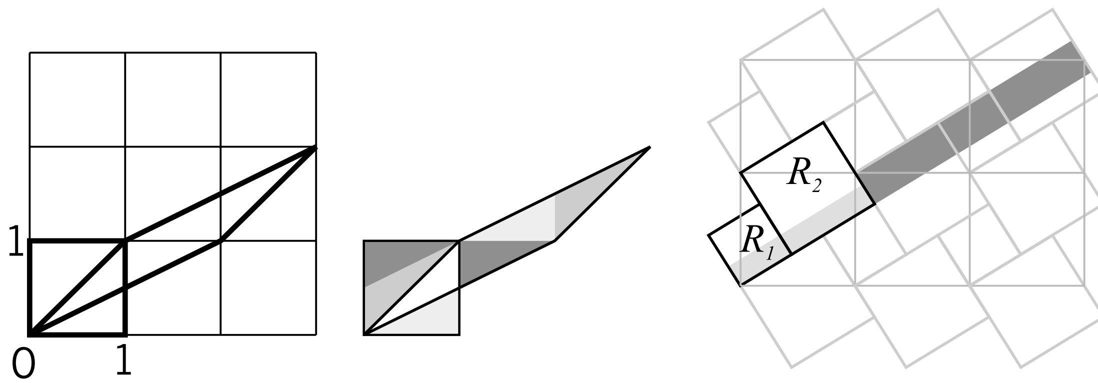

Figure 3: The Cat Map is a toral automorphism. (a) The operation of the linear map on the unit square. (b) Under the mod operation, the image is exactly the unit square. (c) Tesselation by rectangles R_1 and R_2 forms an infinite partition on R^2. However, since the map is defined on the toral space T^2, only two rectangles are required to cover the space. The filled gray boxes illustrate that R_1 and R_2 are mapped completely across a union of rectangles.

4 Applications of Markov Partitions

In addition to the establishing the link to Symbol Dynamics, the *Markov Partition* has another direct application in the 1-dimensional case. In a dynamical system, we are often interested in the overall behavior of the map — the evolution of an ensemble of initial conditions. The Frobenius-Perron operator is used to describe this evolution. When the map is *Markov*, this operator reduces to finite dimensional stochastic *transition matrix*. Following the same development as in probability theory, the stationary (invariant) density associated with these maps is described by the eigenvector for the eigenvalue 1. If the system meets certain ergodic conditions, this density will describe the time average behavior of the system.

The analysis of the ensemble behavior of a dynamical system via its transition matrix is such a powerful tool that we would like to apply it to other 1-dimensional systems, even when they may not be Markov. A general technique for approximating the invariant density of a map is called *Ulam's Method*, conjectured by Ulam [9] in 1960 and later proven by Li [8] in 1976. The method relies upon the fact that Markov maps are dense in function space. Consult [10] for a thorough description of this technique.

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