Synchronization in random weighted directed networks

Maurizio Porfiri, Daniel J. Stilwell and Erik M. Bollt

Abstract— **We assess synchronization of oscillators that are coupled via a time-varying stochastic network, modeled as a weighted directed random graph that switches at a given rate between a set of possible graphs. The existence of any graph edge is probabilistic and independent from the existence of any other edge. We further allow each edge to be weighted differently. Even if the network is always instantaneously not connected, we show that sufficient information is propagated through the network to allow almost sure synchronization as long as the expected value of the network is connected, and that the switching rate is sufficiently fast.**

Keywords: synchronization, random graphs, directed graphs, weighted graphs, fast switching, ad hoc networking

I. Introduction

Since Huygen's early observations of weakly coupled clock pendula [20], synchronization has been found in a wide variety of phenomena, especially in recent years, ranging from biological systems that include fire flies in the forest [7], [29], animal gates [9], descriptions of the heart [15], [19], [43], and improved understanding of brain seizures [31], to chemistry [23], nonlinear optics [33], and meteorology [11]. Many excellent reviews now available, including [8], [42], [14], [30], [37].

Despite the very large literature to be found on synchronization, the great majority of research activities have been focused on static networks whose connectivity and coupling strengths are constant in time. For example, static networks are assumed for the analysis of [3], [35], [36]. However, there are applications where the coupling strengths and even the network topology can evolve in time. Recent work such as [21], [26], [27], [39], [41], [46], [4], [5] are amongst the few to consider time-dependent couplings. There is even less previous work concerning problems of stochastic time-varying couplings as we address in this paper, see e.g. [6], [10], [32].

For the case of static networks, extensive research activity has led to many well-established and broad criteria for oscillators synchronization, see e.g. [35], [36]. In [40] it is shown that if the time-varying network topology has a uniform time-average and switches at a sufficiently fast rate, then synchronization is asymptotically achieved if the static time-average of the network topology asymptotically supports synchronization. In this case, the concept of fast switching enables the analysis of a time-varying network to be based on a static time-average of the network. In particular, the result applies to periodically switching network topologies.

The principal contribution of this paper is to extend the deterministic result of [40] to time-varying networks whose topology changes randomly over time. We formulate the synchronization problem over a random communication network, where each communication link has a different probability to exist and communication channels are arbitrary weighted. We show that if the average network asymptotically supports synchronization, then synchronization is achieved asymptotically almost surely if the network topology switches at a sufficiently fast rate. This result is obtained by using a mixture of findings from stochastic stability theory, see e.g. [24], linear system theory, see e.g. [38], and fast switching theory, see e.g. [22].

Our present work is motivated by a variety of emerging applications, including cooperative control [13] and [18], mathematical epidemiology [39], mobile ad-hoc networks [28], and opinion dynamics [17]. In all of these cases, information is shared along agents via a time-varying network topology that may change randomly. In the context of mathematical epidemiology, our results relate to models of how disease propagates through large network of moving agents. Such a network would be time-varying since infections depend on proximity among infected and susceptible moving agents.

The paper is organized as follows. In Section II we review the concept of random graph and we formally define the synchronization problem. In Section III we derive some preliminary results on stochastic linear systems needed for proving our main claim which appears in Section IV. In Section V our main result is illustrated by a numerical simulation for a network of Rössler oscillators. Section VI is left for conclusions.

Our notation throughout is standard. $\|\cdot\|$ refers to the Euclidean norm in $\mathbb{R}^N$ or corresponding induced norm. The vector in $\mathbb{R}^N$ that consists of all unit entries is denoted $e = [1, \ldots, 1]^{\mathrm{T}}$, and $\otimes$ is the standard Kronecker product. $\mathbb{Z}^+$ refers to the set of nonnegative integers, also I_N is the $N \times N$ identity matrix.

II. Problem Statement

A. Random Graph

For a random directed graph G with N vertices, the existence of an edge from vertex i to vertex $j \neq i$ in the set

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$V = \{1, ..., N\}$ is determined randomly and independently of other edges with probability $p_{ij} \in [0, 1]$. We assume that the graph does not have loops. That is, no single edge starts and ends at the same vertex. In addition, we introduce the weight matrix $W = [w_{ij}]$, where $w_{ii} = 0$ for $i \in V$. The adjacency matrix $A = [a_{ij}]$ of a directed weighted random graph is a random matrix with all zeros on the main diagonal, and off-diagonal elements defined as:

$$a_{ij} = \begin{cases} w_{ij} \text{ with probability } p_{ij} \\ 0 \text{ with probability } 1 - p_{ij} \end{cases}$$

for $i \neq j$. For the case that all p_{ij} are equal to p for every $i, j \in V$, the random graph is the well known Erdös-Rényi graph [12], for which the question of percolation transitions is well understood [1]. That is, the question of when a giant component develops above a critical value of p, allowing a path between most vertices. The present approach should allow for specific popular network configurations, including small world graphs [44], and scale-free graphs [2]. The out-degree matrix $D = \mathrm{diag}(d)$ is a random matrix whose nonzero elements are $d_i = \sum_{j=1}^{n} a_{ij}$. The Laplacian matrix $L = D - A$ is defined as the difference between the adjacency matrix A and the out-degree matrix D. The finite sample space of the random graph is indicated by $\mathcal{G}$, and the elementary events (possible graphs) are indicated by $G^{(j)}$, $j = 1, ..., |\mathcal{G}|$ where $|\cdot|$ denotes cardinality. The graph Laplacian corresponding to $G^{(j)}$ is $L^{(j)}$.

The Laplacian matrix $L = [l_{ij}]$ is a zero row-sum matrix, and therefore the null space of L contains $e = [1, .., 1]^{\mathrm{T}}$. Properties of directed weighted graphs may be found in the comprehensive paper [45].

Since the graph edges are independent random variables, the expected value of the graph Laplacian, written $E[L] = [E[l_{ij}]]$, may be computed entrywise by

$$E[l_{ij}] = \begin{cases} -p_{ij} w_{ij}, & i \neq j \\ + \sum_{k=1}^{N} p_{ik} w_{ik}, & i = j \end{cases} \tag{1}$$

The matrix $E[L]$ corresponds to a weighted directed graph which does not necessarily belong to $\mathcal{G}$. We refer to this graph as the *average graph*, denoted $E[G]$.

If each edge has the same probability to exist, $p_{ij} = p \neq 0$, as in the Erdös-Rényi graph, then the average graph Laplacian $E[L]$ is equal to the graph Laplacian of a fully connected graph multiplied by p, where a fully connected graph has a weighted directed edge from each node to every other node.

B. Synchronization Problem

We consider a dynamic system consisting of N identical oscillators interconnected pairwise via a stochastic, weighted, unidirectional information network

$$\dot{x}_i(t) = f(x_i(t)) + \sigma B(t) \sum_{j=1}^{N} l_{ij}(t) x_j \tag{2}$$
$$x_i(0) = x_{i0}, \ i = 1, \dots N, \ t \geq 0$$

where $x_i(t) \in \mathbb{R}^n$ is the random state of oscillator i, $x_{i0} \in \mathbb{R}^n$ is its initial condition, $f : \mathbb{R}^n \to \mathbb{R}^n$ describes the oscillators' individual dynamics, $B(t) \in \mathbb{R}^{n \times n}$ models coupling between agents, $\sigma > 0$ is a control parameter that partially assigns coupling strength between oscillators, and scalars $l_{ij}(t)$ are the element of the time-varying graph Laplacian $L(t)$. We collect all the states of the system in the nN dimensional vector $x(t) = [x_1(t), \dots, x_N(t)]^{\mathrm{T}} \in \mathbb{R}^{nN}$.

The network topology $G(t)$ characterized by the corresponding graph Laplacian $L(t)$ switches at a series of time instants $\{\Delta k | k \in \mathbb{Z}^+\}$, where Δ is a fixed period. The set of equations (2) describes a nonlinear stochastic switched system. During each time interval $[k\Delta, (k+1)\Delta)$ with $k \in \mathbb{Z}^+$, the communication network $G(t)$ is constant and equals the random graph G_k. We assume that the G_k's are independent equally distributed random graphs as described in Subsection II-A.

We say that the system of oscillators is synchronized if the state vectors for all oscillator are identical. Specifically, the oscillators are synchronized if $x(t) = e \otimes s(t)$ for some $s(t) \in \mathbb{R}^n$ that is a solution of the individual oscillator

$$\dot{s}(t) = f(s(t)) \tag{3}$$

We refer to the the manifold in $\mathbb{R}^{nN}$ consisting of trajectories $e \otimes s(t)$ where $s(t)$ is a solution of (3) as the *synchronization manifold*. Note also that the range of $e \otimes I_n$ contains the synchronization manifold.

C. Decomposition

Synchronization of the system of oscillators can be assessed by examining local stability with respect to the synchronization manifold. Linearizing the system of oscillators about the trajectory $e \otimes s(t) \in \mathbb{R}^{nN}$ on the synchronization manifold yields,

$$\dot{z}(t) = (I_N \otimes F(t) + \sigma L(t) \otimes B(t)) z(t) \tag{4}$$

where $z(t) = x(t) - e \otimes s(t)$. To analyze asymptotic stability of the set of oscillators, we decompose the state of (4) into a component that evolves along the synchronization manifold, and a component that evolves transverse to the synchronization manifold. For analysis, it suffices to show that the component that evolves transverse to the synchronization manifold asymptotically approaches the synchronization manifold.

Decomposition of the oscillator states is based on a matrix $W \in \mathbb{R}^{N \times (N-1)}$ that satisfies $W^{\mathrm{T}} e = 0$ and $W^{\mathrm{T}} W = I_{N-1}$. Note that the vector of synchronized oscillator states $\mathrm{e} \otimes s(t)$ is in the null space of $W^T \otimes I_n$ and in the range of $e \otimes I_n$. The state vector $z(t)$ in (4) can be decomposed as

$$z(t) = (W \otimes I_n)\zeta(t) + e \otimes z_s(t)$$

where

$$\zeta = (W \otimes I_n)^{\mathrm{T}} z$$

and

$$z_s = \frac{1}{N}((e \otimes I_n)^{\mathrm{T}} z)$$

Note that $z_s(t)$ is the average of all the components in $z(t)$, and that $((W \otimes I_n)\zeta)^{\mathrm{T}}(e \otimes z_s) = 0$. Using the state transformation

$$\begin{bmatrix} z_s(t) \\ \zeta(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{N}(e \otimes I_n)^{\mathrm{T}} \\ (W \otimes I_n)^{\mathrm{T}} \end{bmatrix} z(t)$$

the linearized oscillator dynamics (4) are partitioned

$$\dot{z}_s(t) = F(t)z_s(t) + \sigma(e^{\mathrm{T}}L(t)W \otimes B(t))\zeta(t) \tag{5a}$$

$$\dot{\zeta}(t) = (I_{N-1} \otimes F(t) + \sigma W^{\mathrm{T}}L(t)W \otimes B(t))\zeta(t) \tag{5b}$$

Thus local almost sure asymptotic synchronization of the set of oscillators (2) is achieved if $\zeta(t)$ in (5b) almost surely converges to zero, see e.g. [25]. The definition of almost sure convergence may be found, for example, in [16] (Chapter 5).

III. Preliminary Propositions

In this section we present two propositions on linear system stability which are used to determine sufficient conditions for almost sure asymptotic synchronization of (2). The first proposition establishes the relationship between asymptotic stability of a linear switched time-varying system and that of a derived sample-data system.

Proposition 1: Consider the stochastic linear system

$$\dot{y}(t) = (A(t) + B(t) \otimes H(t))y(t), \ t \geq 0 \tag{6}$$

where $y(t) \in \mathbb{R}^m$, $A(t) \in \mathbb{R}^{m\times m}$ and $B(t) \in \mathbb{R}^{m\times m}$ are bounded and continuous functions for all $t \geq 0$, and $H(t) \in \mathbb{R}^{m\times m}$ is a bounded random process that for some $\Delta > 0$ is constant for all $t \in [k\Delta, (k+1)\Delta)$ and switches at time instants Δk, for all $k \in \mathrm{Z}^+$. Let $y_k \equiv y(k\Delta)$. If $y_k \overset{a.s.}{\to} 0$, then $y(t) \overset{a.s.}{\to} 0$.

Proof: For any $t \in [k\Delta, (k+1)\Delta)$, $y(t) = \Phi(t, k\Delta)y_k$ where Φ is the transition matrix of the linear system (6). Since A, B and H are bounded, there exist positive constants constants α, β and η such that for any $t \geq 0$,

$$\|A(t)\| \leq \alpha, \quad \|B(t)\| \leq \beta, \quad \|H(t)\| \leq \eta \tag{7}$$

where $\|\cdot\|$ is the induced Euclidean norm. From the Gronwall-Belmann inequality, (see e.g. Exercise 3.12 in [38])

$$\|y(t)\| \leq \|y_k\| \exp \int_{k\Delta}^{(k+1)\Delta} (\|A(\tau) + B(\tau) \otimes H(\tau)\|)dt$$

Thus for any $t \in [k\Delta, (k+1)\Delta)$

$$\|y(t)\| \leq \exp((\alpha + \beta\eta)\Delta)\|y_k\|$$

and claim follows. ■

For clarity we restate the well-known Borel-Cantelli Lemma in the form presented in Lemma 1 of [24] (Chapter 8).

Lemma 1: Consider the stochastic process $X_k \in \mathbb{R}^m$. If $f : \mathbb{R}^m \to \mathbb{R}$ is a nonnegative function and

$$\sum_{k=0}^{\infty} E[f(X_k)] < \infty$$

then $f(X_k) \overset{a.s.}{\to} 0$.

The following proposition generalizes the claim in Theorem 8 of [24] (Chapter 8) to time-varying systems.

Proposition 2: Consider the stochastic system

$$X_{k+1} = A_k X_k \tag{8}$$

where the A_k are mutually independent random matrices and $X_0 \in \mathbb{R}^m$. Suppose there is a sequence of symmetric positive semidefinite matrices Q_k, such that for any k

$$E[A_k^{\mathrm{T}} Q_{k+1} A_k] - Q_k = -C \tag{9}$$

where E indicates expected value, and C is symmetric positive definite, then (8) is almost surely asymptotically stable in the sense that $X_k \overset{a.s.}{\to} 0$ for any initial condition X_0.

Proof: Our proof is similar to that of Theorem 1 in [24] (Chapter 8). Define the quadratic Lyapunov function $v_k(x) = \frac{1}{2}x^{\mathrm{T}}Q_k x$, $x \in \mathbb{R}^m$ and the related random process

$$V_k = v_k(X_k) = \frac{1}{2}X_k^{\mathrm{T}} Q_k X_k \tag{10}$$

Thus, using iteratively (8), (9) and (10) we obtain

$$E[V_1] = V_0 - \frac{1}{2}X_0^{\mathrm{T}} C X_0$$

$$E[V_2|X_1] = v_1(X_1) - \frac{1}{2}X_1^{\mathrm{T}} C X_1$$

$$E[V_k|X_{k-1}, \ldots X_1] = v_{k-1}(X_{k-1}) - \frac{1}{2}X_{k-1}^{\mathrm{T}} C X_{k-1}$$

$$E[V_k|X_1, \ldots X_{k-1}] = V_0 - \frac{1}{2}\sum_{h=0}^{k-1} X_h^{\mathrm{T}} C X_h \tag{11}$$

Since C and Q_k are positive semidefinite, (11) yields

$$V_0 \geq V_0 - E[V_k] = \frac{1}{2}\sum_{h=0}^{k-1} E[X_h^{\mathrm{T}} C X_h] \tag{12}$$

Thus the right-hand side of (12) is bounded above. Since C is positive semi-definite, the right-hand side of (12) is non-negative, and direct application of Lemma 1 with $f(X_k) = 1/2X_k^{\mathrm{T}} C X_k$ yields $X_k \overset{a.s.}{\to} 0$. ■

IV. Synchronization through fast-switching

Our principal result is to show that the network of coupled oscillators with stochastic time-varying $L(t)$ can synchronize even if the network is insufficiently coupled to support synchronization at every instant of time. We show that synchronization is achieved as long as the average network $E[G]$, corresponding to the graph Laplacian $E[L]$ in (1), supports synchronization and the switching period Δ between new samples of the network topology is sufficiently small. In other words, analysis of synchronization for the stochastic time-varying network (2) reduces to analysis of synchronization for the static network

$$\dot{x}_i(t) = f(x_i(t)) + \sigma B(t)\sum_{j=1}^{N} E[l_{ij}]x_j, \ i = 1, \ldots N, \ t \geq 0 \tag{13}$$

along with sufficiently fast switching of the network topology. The case of static networks has been extensively addressed in the literature, and analysis tools include the well-known master-stability function [35].

Theorem 1: Consider the deterministic dynamic system

$$\dot{y}(t) = (I_{N-1} \otimes F(t) + \sigma W^{\mathrm{T}} E[L] W \otimes B(t)) y(t) \tag{14}$$

representing the linearized transverse dynamics of (13). Assume that $F(t)$ and $B(t)$ are bounded and continuous for all $t \geq 0$. If (14) is uniformly asymptotically stable, there is a time-scale $\Delta^* > 0$ such that for any shorter time-scale $\Delta < \Delta^*$ the stochastic system (2) asymptotically synchronizes almost surely.

Proof:

We define $M(t) \equiv (I_{N-1} \otimes F(t) + \sigma W^{\mathrm{T}} L(t) W \otimes B(t))$ and rewrite the variational equation (5b) as

$$\dot{\zeta}(t) = M(t)\zeta(t) \tag{15}$$

Consequently equation (14) my be compactly rewritten as

$$\dot{y}(t) = \bar{M}(t) y(t) \tag{16}$$

where $\bar{M}(t) \equiv (I_{N-1} \otimes F(t) + \sigma W^{\mathrm{T}} E[L] W \otimes B(t))$. With Proposition 1 in mind, we define the sequences $\zeta_k \equiv \zeta(k\Delta)$ and $y_k \equiv y(k\Delta)$, for $k \in \mathbb{Z}^+$. For any $k \in \mathbb{Z}^+$

$$\zeta_{k+1} = \Gamma_k \zeta_k \tag{17a}$$

$$y_{k+1} = \Theta_k y_k \tag{17b}$$

where Γ_k and Θ_k are the transition matrices of (15) and (16) over the time interval $[k\Delta, (k+1)\Delta)$, respectively. By hypothesis, y_k in (17b) asymptotically converges to zero for any initial condition y_0. Our task is to show that ζ_k in (17a) converges to zero almost surely for any initial condition if Δ is sufficiently small.

Since F and B are bounded for all $t \geq 0$, there are positive constants ϕ and β such that for any $t \geq 0$, $\|F(t)\| \leq \phi$ and $\|B(t)\| \leq \beta$. Since the event sample space is finite $\|L(t)\| \leq \lambda$, where $\lambda = \max_{j=1,\ldots|\mathcal{G}|} \|L^{(j)}\|$. By definition $\|W\| = 1$, therefore for any $t > 0$

$$\|M(t)\| \leq \alpha, \qquad \|\bar{M}(t)\| \leq \alpha \tag{18}$$

where $\alpha = \phi + \sigma\beta\lambda$.

Since (16) is uniformly asymptotically stable and $\bar{M}(t)$ is continuous there exists symmetric continuously differentiable matrix $Q(t)$ such that (see e.g. Theorem 7.2 of [38])

$$\eta I \leq Q(t) \leq \rho I \tag{19}$$

The related Lyapunov function $v(t,y) = \frac{1}{2} y^{\mathrm{T}} Q(t) y$ satisfies $\frac{d}{dt} v(t, y(t)) \leq -\mu \|y(t)\|^2$, where η, ρ and μ are positive scalars. For linear systems, uniform asymptotic stability is equivalent to uniform exponential stability, thus in $[k\Delta, (k+1)\Delta)$, (see e.g. the proof of Theorem 7.4 of [38])

$$v((k+1)\Delta, y_{k+1}) - v(k\Delta, y_k) \leq (1 - \exp(-\mu\Delta/\rho)) \|y_k\|^2 \tag{20}$$

From (20) the following condition arises

$$\Theta_k^{\mathrm{T}} Q((k+1)\Delta) \Theta_k - Q(k\Delta) = -P \tag{21}$$

where P is a symmetric positive definite matrix satisfying

$$\|P\| \geq (1 - \exp(-\mu\Delta/\rho)) \tag{22}$$

In addition, since $\bar{M}(t)$ bounded, from the Gronwall-Belmann inequality, see e.g. Exercise 3.12 in [38], we have

$$\|\Theta_k\| \leq \exp \int_{k\Delta}^{(k+1)\Delta} \|\bar{M}(t)\| dt$$

which yields

$$\|\Theta_k\| \leq \exp(\alpha\Delta) \tag{23}$$

Recalling the Peano-Baker expansion for Γ_k (see e.g. Chapter 3 of [38])

$$\Gamma_k = I_{(N-1)n} + \int_{k\Delta}^{(k+1)\Delta} M(\sigma_1) d\sigma_1 + \sum_{i=2}^{\infty} \int_{k\Delta}^{(k+1)\Delta} M(\sigma_1) \int_{k\Delta}^{\sigma_1} \cdots \int_{k\Delta}^{\sigma_{i-1}} M(\sigma_i) d\sigma_i \cdots d\sigma_1$$

we can express

$$R_k = \Gamma_k - \Theta_k \tag{24}$$

as

$$R_k = \int_{k\Delta}^{(k+1)\Delta} (M(\sigma) - \bar{M}(\sigma)) d\sigma + \varepsilon_k \tag{25}$$

where

$$\varepsilon_k = \sum_{i=2}^{\infty} \int_{k\Delta}^{(k+1)\Delta} (M(\sigma_1) - \bar{M}(\sigma_1)) \int_{k\Delta}^{\sigma_1} \cdots \int_{k\Delta}^{\sigma_{i-1}} (M(\sigma_i) - \bar{M}(\sigma_i)) d\sigma_i \cdots d\sigma_1 \tag{26}$$

By taking the expected value of both sides of (25) we have

$$E[R_k] = E[\varepsilon_k] \tag{27}$$

From (18), the norm of the first term on the RHS of (25) is bounded by $2\Delta\alpha$. On the other hand from (18) and (26) the norm of ε_k may be bounded by

$$\|\varepsilon_k\| \leq \sum_{i=2}^{\infty} (2\Delta\alpha)^i = \exp(2\Delta\alpha) - 1 - 2\Delta\alpha \tag{28}$$

which implies

$$\|E[\varepsilon_k]\| \leq \exp(2\Delta\alpha) - 1 - 2\Delta\alpha \tag{29}$$

Using (18) and (29), the norm of R_k is bounded by

$$\|R_k\| \leq \exp(2\Delta\alpha) - 1 \tag{30}$$

We emphasize that from (28) and (30),

$$\|R_k\| \leq O(\Delta), \qquad \|\varepsilon_k\| \leq O(\Delta^2) \tag{31}$$

Next, we show that for sufficiently small values of Δ, the matrix $Q_k = Q(k\Delta)$ defines a quadratic Lyapunov function for the system (5b) in the sense of Proposition 2. Indeed by using substituting (24) into (21) and by using (27), we obtain

$$\begin{aligned} E[\Gamma_k^{\mathrm{T}} Q_{k+1} \Gamma_k] - Q_k = \\ E[(R_k + \Theta_k)^{\mathrm{T}} Q_{k+1} (R_k + \Theta_k)] - Q_k = \\ -P + \Theta_k^{\mathrm{T}} Q_{k+1} E[\varepsilon_k] + E[\varepsilon_k]^{\mathrm{T}} Q_{k+1} \Theta_k + E[R_k^{\mathrm{T}} Q_{k+1} R_k] \end{aligned} \tag{32}$$

Combining all the bounds in (18), (19), (22), (23), (29) and (30), (32) yields

$$E[\Gamma_k^{\mathrm{T}} Q_{k+1} \Gamma_k] - Q_k \leq -g(\Delta) I_{(N-1)n}$$

where the continuous function $g(\Delta)$ is defined by

$$g(\Delta) = (1 - \exp(-\mu\Delta/\rho)) - 2\exp(\alpha\Delta)\rho \\ (\exp(2\Delta\alpha) - 1 - 2\Delta\alpha) - \rho(\exp(2\Delta\alpha) - 1)^2$$

Since $g(0) = 0$, $\frac{d}{d\Delta} g(0) = \frac{\mu}{\rho} > 0$, and $g(\Delta) \to -\infty$ as $\Delta \to \infty$, there exists Δ^* such that $g(\Delta) > 0$ for all $\Delta \in (0, \Delta^*)$ and by applying Proposition 2 the claim follows. ■

V. Illustration

To illustrate fast switching concepts applied to synchronization of a set of oscillators coupled through a randomly switching graph, we consider a set of r Rössler attractors

$$\dot{x}_i(t) = -y_i(t) - z_i(t) - \sigma \sum_{j=1}^{r} l_{ij} x_j(t)$$
$$\dot{y}_i(t) = x_i(t) + a y_i(t)$$
$$\dot{z}_i(t) = b + z_i(t)(x_i(t) - c)$$

where $i = 1, \ldots, r$, $a = 0.165$, $b = 0.2$, $c = 10$, and $\sigma = 0.3$. Oscillators are coupled through the x_i variables via l_{ij} which changes randomly at the transition instants $k\Delta$. Each edge is assigned the same probability to exists, $p_{ij} = p = 0.05$ and the same unitary weight. One may naturally ask if such a directed random graph is close to percolation, since it is closely related to the undirected Erdös-Rényi graph [12] for which the percolation question can be directly addressed. However, the percolation problem is not relevant here, since it is not necessary for our network to be instantaneously connected. Indeed, it is possible for the network to instantaneously consist of many subcomponents when the system is changed with a fast enough Δ. We consider as example, a network of 25 oscillators and we assume that switching occurs with a period $\Delta = 1$. Figure 1 and Figure 2 depict the x coordinate of the set of coupled Rössler oscillators using the average graph and the random graph respectively, versus the time t. Figure 3 reports the dynamics of the network topology for the first 6 time intervals.

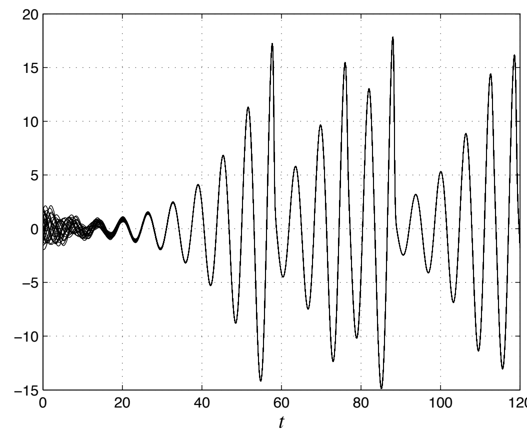

Fig. 1. Time evolution of the x coordinate of the set of coupled Rössler oscillators using the average graph

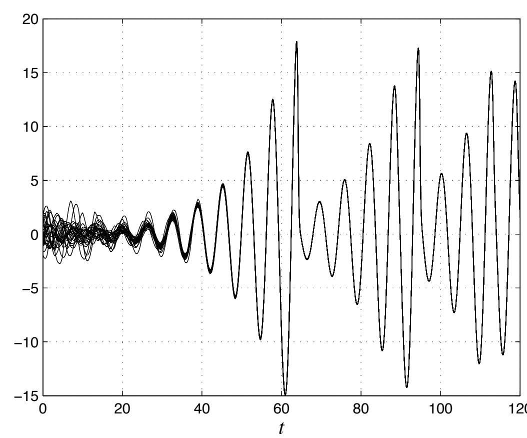

Fig. 2. Time evolution of the x coordinate of the set of coupled Rössler oscillators using the random graph

VI. Conclusions

New generalizations on synchronization of mutually coupled oscillators are presented. We pose the synchronization problem in a stochastic framework where communication among nodes is modeled as a weighted directed random switching graph. We utilize tools based on fast switching and stochastic stability, and show that synchronization is asymptotically achieved if the average communication network asymptotically supports synchronization and if the network topology is changing with a sufficiently fast rate. A numerical simulation illustrates the theoretical achievements of the present paper.

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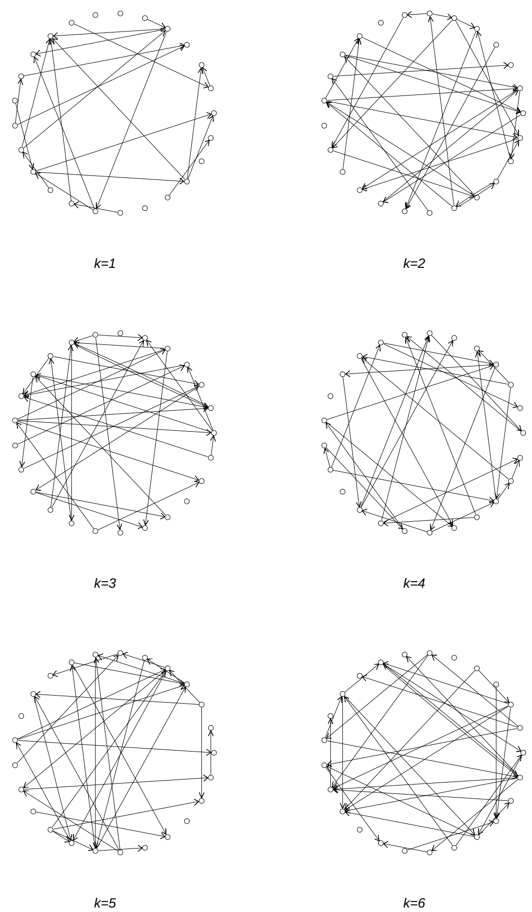

Fig. 3. Dynamics of the random network

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