PROBABILITY DENSITY FUNCTIONS OF SOME SKEW TENT MAPS

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Abstract. We consider a family of chaotic skew tent maps. The skew tent map is a two-parameter, piecewise-linear, weakly-unimodal, map of the interval $F_{a,b}$. We show that $F_{a,b}$ is Markov for a dense set of parameters in the chaotic region, and we exactly find the probability density function (pdf), for any of these maps. It is well known, [1], that when a sequence of transformations has a uniform limit F, and the corresponding sequence of invariant pdf's has a weak limit, then that invariant pdf must be F invariant. However, we show in the case of a family of skew tent maps that not only does a suitable sequence of convergent sequence exist, but they can be constructed entirely within the family of skew tent maps. Furthermore, such a sequence can be found amongst the set of Markov transformations, for which pdf's are easily and exactly calculated. We then apply these results to exactly integrate Lyapunov exponents.

1. Introduction

Let $F_{a,b}$ denote the two-parameter piecewise-linear map on the interval $[0,1]$ satisfying

$$F_{a,b}(x) = \begin{cases} b + \frac{1-b}{a}x & \text{if } 0 \le x < a \\ \frac{1-x}{1-a} & \text{if } a \le x \le 1 \end{cases} \tag{1.1}$$

with $0 < a < 1$ and $0 \le b \le 1$. It has been shown that [2] in the following region of the parameter space,

$$D = \{(a,b) : b < 1/(2-a) \text{ and } ((b < 1-a) \text{ or } (b \ge 1-a \text{ and } b > a))\}, \tag{1.2}$$

$F_{a,b}$ has chaotic dynamics, and in the parameter subset

$$D_0 = \{(a,b) : b < 1/(2-a) \text{ and } (b < 1-a)\}, \tag{1.3}$$

chaotic dynamics occur on the entire interval $0 \le x \le 1$. See Bassein [2] for a complete classification of the dynamics in parameter space, $(a,b) \in (0,1) \times [0,1]$.

We will show that $F_{a,b}$ is Markov for a dense set of (a,b) in D_0. This is interesting because the probability density function for any Markov transformation in this set can be found exactly to be a piecewise constant function. We can use these exact results to approximate the probability density function for any other transformation in D_0. We also show that Lyapunov exponents can be calculated exactly on the Markov set, and therefore, efficiently approximated on all of D_0.

In Section 2 we review a sufficient condition for a piecewise-linear map on the interval to be Markov, and we discuss symbolic dynamics for this type of map. Then, we prove that Markov maps are dense in the parameter set D_0. Illustrating the simplifications of Markov maps, we discuss in Section 3 the techniques used to exactly find the probability density function and the Lyapunov exponent. We conclude the discussion of Markov maps with an explicit example in Section 4. In Section 5 we finish with a proof that the properties of the remaining maps in the parameter set D_0 can be approximated sufficiently by members of the dense set of Markov maps.

We chose the form of $F_{a,b}$ from [2], wherein can be found a concise discussion of the full range of topological dynamics in the parameter space $(a,b) \in (0,1) \times [0,1]$. The form of the skew tent map we study, equation (1.1), is topologically conjugate to the skew tent map in the topological studies of Misiurewicz and Visinescu, [3], [4]. Piecewise linear transformations on the interval have been widely studied, and under different names

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as well: broken linear transformations [5], weak unimodal maps [6]. Other closely related areas in the study the chaotic behavior of these maps include the stability of the associated Frobenius-Perron operator [7], the topological entropy [4], and kneading sequences [8].

2. Markov Transformations

In this section, we classify the Markov partitions which are prominent in calculations in the next section. In the special, but important, case that a transformation of the interval is Markov, the symbol dynamic is simply presented as a finite directed graph. A Markov transformation is defined as follows.

Definition 2.1 ([1]). *Let $I=[c,d]$ and let $\tau : I \to I$. Let $\mathcal{P}$ be a partition of I given by the points $c=c_0<c_1<\ldots<c_p=d$. For $i=1,\ldots,p$, let $I_i=(c_{i-1},c_i)$ and denote the restriction of τ to I_i by τ_i. If τ_i is a homeomorphism from I_i onto some connected union of intervals of $\mathcal{P}$, then τ is said to be Markov. The partition $\mathcal{P}$ is said to be a Markov partition with respect to the function τ.*

The following result describes a set in D_0 for which $F_{a,b}$ is Markov.

Theorem 2.2. *For a given $(a,b)\in D_0$, if $x_0=1$ is a member of a periodic orbit, then $F_{a,b}$ is Markov.*

Proof. Set $F=F_{a,b}$. Assume $x_0=1$ is a member of a period n orbit $(n>1)$. Next, form a partition of $[0,1]$ using the n members of the periodic orbit. The two endpoints of the interval are included since $\forall(a,b)\in D_0$, $F(1)=0$. Order these n points so $0=c_0<c_1<\ldots<c_{n-1}=1$, regardless of the iteration order. For $i=1,\ldots,n-1$, let $I_i=(c_{i-1},c_i)$ and denote the restriction of F to I_i by F_i.

For a given $I_i=(c_{i-1},c_i)$, the endpoints c_{i-1} and c_i will map exactly to two members of the partition endpoints by definition of the periodic orbit. Let these points be c_j and c_k, with $c_j<c_k$ and $j,k\in\{0,\ldots,n-1\}$. The only turning point of the map is $x=a$, and $\forall(a,b)\in D_0$, $F(a)=1$. Therefore $x=a$ must always be part of the period orbit and a member of the partition endpoints, implying each F_i is linear and hence, a homeomorphism. Also $F_i(I_i)=(c_j,c_k)$, a connected union of intervals of the partition. By definition, F is Markov. □

At this point, symbolic notation becomes useful. The point $x=a$ is the critical point at the "center" of the interval, denoted by the letter C. All $a<x\le 1$ is right of a, represented by R, and all $0\le x<a$ is left of a, represented by L. Represent each step of the iteration map by one of these three symbols. All parameter sets for which $F(x)$ is Markov must have a period-n orbit containing the point $x=a$ and be of the form $\left\{\overline{a,\ldots,F^{n-1}(a)},\ldots\right\}$. For example, the period three orbit has the form $\left\{\overline{a,1,0},\ldots\right\}$ and occurs for any parameter set (a,b) on the curve $b=a$. It repeats the pattern CRL, which we call the kneading sequence $K(F_{a,b})=(CRL)^\infty$.

If a periodic orbit contains the point $x=a$ and $a\neq b$, the point $x=b$ will either be greater or less than a. For a period four orbit with $a>b$, the symbolic sequence must repeat CRLL, or if $a<b$, CRLR. Therefore, a period four is found two ways. See Figure 1 for a cobweb diagram of an example period 4 CRLL orbit.

Repeating this method for period five and higher, we see that there are 2^{n-3} possible combinations of the C-L-R sequences for a period n orbit which includes the critical point. The exponent $n-3$ reflects the necessary 3-step prefix CRL. A full binary tree with 2^{n-3} leaves on each tier is possible, each implying a condition on the parameters and forming a countable set of curves in parameter space [4]. We can now restate Theorem 2.2 in terms of kneading sequences.

Corollary 2.3. *If $K(F_{a,b})$ is periodic, then $F_{a,b}$ is Markov.*

We now prove that the function $F_{a,b}(x)$ (1.1) is Markov on a dense set of curves in D_0 (1.3). Define Σ_2 as the space of symbol sequences containing the full family of kneading sequences for two symbols. Define the kneading sequence $\sigma=\sigma_0\sigma_1\sigma_2\ldots$, the metric $d(\sigma,\hat\sigma)=\sum_{i=0}^{\infty}\frac{|\sigma_i-\hat\sigma_i|}{2^i}$, and the norm $||\sigma||=\sum_{i=0}^{\infty}\frac{\sigma_i}{2^i}=\sigma_0+\frac{\sigma_1}{2}+\frac{\sigma_2}{2^2}+\ldots$.

Lemma 2.4. *Periodic σ are dense in Σ_2.*

Proof. $\forall \varepsilon > 0$ and given WOLOG $\phi = \phi_0\phi_1\phi_2\ldots$ which is not periodic, $\exists N > 0$ large enough so that $\sigma_i = \phi_i$, $i \leq N$ and $\sigma = (\phi_0\phi_1\ldots\phi_N)^\infty$ for $||\sigma - \phi||_{\Sigma_2} < \varepsilon$. □

3. Probability density functions and Lyapunov exponents

Another benefit of the set of Markov chaotic functions is that the probability density functions can be determined quite easily. In fact, expanding piecewise linear Markov transformations have piecewise constant invariant probability density functions.

Theorem 3.1 ([1], Theorem 9.4.2)**.** *Let $\tau : I \to I$ be a piecewise linear Markov transformation such that for some $k \geq 1$,*

$$|(\tau^k)'| > 1,$$

wherever the derivative exists. Then τ admits an invariant [probability] density function which is piecewise constant on the partition $\mathcal{P}$ on which τ is Markov.

Using the Frobenius-Perron operator P, the fixed-point function h satisfies the definition $P_F h = h$, implying that h is the probability density function for a measure that is invariant under F. Since F is a piecewise monotone function, the action of the operator is simply

$$P_F h(x) = \sum_{z \in \{F^{-1}(x)\}} \frac{h(z)}{|F'(z)|}.$$

The periodic orbit formed by the iteration of $x = a$ forms a partition of the domain $[0,1]$ on which h is piecewise-constant. On each interval I_i, call the corresponding constant $h_i = h|_{I_i}$.

The probability density function admits an absolutely continuous invariant measure on the Markov partition. This measure can be used to find the Lyapunov exponent, and therefore quantify the average rate of expansion or contraction for an interval under iteration. Set $F = F_{a,b}$ for some $(a,b) \in D_0$ and form a partition of $[0,1]$ using the n members of the periodic orbit so $0 = c_0 < c_1 < \ldots < c_{n-1} = 1$. Assume $c_k = a$, for some $k \in \{1, \ldots, n-2\}$. Note $|F'(x)| = \frac{1-b}{a}$ if $x < a$, and $|F'(x)| = \frac{1}{1-a}$ if $x > a$.

$$\begin{aligned} \Lambda_{a,b} &= \int_0^1 \ln|F'(x)|\, h(x)\ dx \\ &= \int_{c_0}^{c_1} \ln|F'(x)|\, h_1\ dx + \ldots + \int_{c_{n-2}}^{c_{n-1}} \ln|F'(x)|\, h_{n-1}\ dx \\ &= \ln\left|\frac{1-b}{a}\right| \sum_{i=1}^{k}(c_i - c_{i-1})h_i + \ln\left|\frac{1}{1-a}\right| \sum_{i=k+1}^{n-1}(c_i - c_{i-1})h_i. \end{aligned} \tag{3.1}$$

4. Following the left branch

As an example, in this section we will derive the probability density function for $F(x)$ when it is Markov and has a periodic orbit of the pattern CRL, CRLL, CRLLL, ..., following the left branch of the binary tree. In the discussion below, $x = a$ is part of a periodic orbit of period $p = n+3$ and n represents the number of L's after the initial CRL in the symbolic representation of that periodic orbit.

$$\{a, 1, 0, b, F(b), F^2(b), \ldots\} \Rightarrow \overbrace{\text{CRL}\underbrace{\text{LLL}\ldots}_{n}}^{p}\ . \tag{4.1}$$

Proposition 4.1. *The general relation of a and b in the period $p = n+3$ in equation (4.1) is,*

$$\left(\frac{1-b}{a}\right)^{n+1} = \frac{1-a}{b}\ . \tag{4.2}$$

Proof. Rename the branches of the piecewise defined function, f_L on $0 \le x < a$ and f_R on $a \le x \le 1$. Using the geometric series expansion, express $f_L^n(b)$ as

$$f_L^n(b) = \sum_{k=0}^{n} b\left(\frac{1-b}{a}\right)^k = \frac{b\left(1-\left(\frac{1-b}{a}\right)^{n+1}\right)}{1-\left(\frac{1-b}{a}\right)}. \tag{4.3}$$

The parameter sets for each of these periodic orbits is found by solving $f_L^n(b) = a$, where the period is $n+3$. By using equation (4.3), we have the implicit relation (4.2). □

It now seems quite natural for the CRL orbit to occur on the parameter set $b = a$. The equations relating a to b for the different periodic orbits form a pattern in parameter space, and limit to $b = 0$.

On these curves in the chaotic region, an invariant density function h can be found in closed form. It is a piecewise defined function on its associated Markov partitions. Therefore, $h(x)$ has the form

$$h(x) = \begin{cases} h_1 & \text{if } 0 < x < b \\ h_2 & \text{if } b < x < F(b) \\ \vdots & \\ h_{p-1} & \text{if } a < x < 1 \end{cases}. \tag{4.4}$$

Then the action of the Frobenius-Perron operator is

$$\begin{aligned} P_f h(x) &= \frac{h(f_L^{-1}(x))}{|f_L'(f_L^{-1}(x))|}\chi_{[a,1]}(x) + \frac{h(f_R^{-1}(x))}{|f_R'(f_R^{-1}(x))|} \\ &= \left(\frac{a}{1-b}\right) h(f_L^{-1}(x))\chi_{[a,1]}(x) + (1-a)h(f_R^{-1}(x)). \end{aligned} \tag{4.5}$$

We use equation (4.5) to construct a system of equations to solve for $h(x)$. This system has the following form:

$$\begin{aligned} h_1 &= (1-a)h_{p-1} \\ h_2 &= \left(\frac{a}{1-b}\right) h_1 + (1-a)h_{p-1} \\ &\vdots \\ h_{p-1} &= \left(\frac{a}{1-b}\right) h_{p-2} + (1-a)h_{p-1}. \end{aligned} \tag{4.6}$$

For a proof, see the geometry of the inverse function. Note the recursion; the constant h_1 depends only on h_{p-1}. From the monotonicity of the inverse of f_L and f_R, each constant, h_k, depends on h_{k-1} and h_{p-1}.

Proposition 4.2. *The system of Equations (4.6) is underdetermined for the variables $h_1, h_2, \dots, h_{p-1}$. The determinant is zero precisely when the parameters satisfy equation (4.2).*

Proof. This is proved by induction. Base case: $n = 0$ and $p = 3$:

$$h_1 = (1-a)h_2 \tag{4.7}$$

$$h_2 = \frac{a}{1-b}h_1 + (1-a)h_2. \tag{4.8}$$

This set of equations can be rewritten as the homogeneous system

$$\begin{pmatrix} 1 & -(1-a) \\ -\frac{a}{1-b} & 1-(1-a) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.9}$$

To find if there is a unique solution, find the determinant M_0:

$$M_0 = \begin{vmatrix} 1 & -(1-a) \\ -\frac{a}{1-b} & 1-(1-a) \end{vmatrix} = 1-(1-a)-(1-a)\frac{a}{1-b}. \tag{4.10}$$

Substituting equation (4.2) for $n=0$, $b-a=0$ implies that $M_0=0$. To prove $M_{j+1}=0$, use the relation

$$M_j = 1-(1-a)\sum_{k=0}^{j}\left(\frac{a}{1-b}\right)^k, \tag{4.11}$$

and assume equation (4.2) is true for $n=j$. Therefore,

$$\begin{aligned} M_{j+1} &= 1-(1-a)\sum_{k=0}^{j+1}\left(\frac{a}{1-b}\right)^k \\ &= 1-(1-a)\left(\frac{1-\left(\frac{a}{1-b}\right)^{j+2}}{1-\left(\frac{a}{1-b}\right)}\right) \\ &= \left(\frac{a(1-a)}{1-a-b}\right)\left(\left(\frac{a}{1-b}\right)^{j+1}-\left(\frac{b}{1-a}\right)\right). \end{aligned} \tag{4.12}$$

Substituting equation (4.2) for $n=j$, $M_{j+1}=0$. □

The freedom implied by Proposition 4.2 is expected since the density function must be normalized. Set the area under the curve to one:

$$\int_0^1 h(x)dx = 1. \tag{4.13}$$

Using the partition, $0=c_0<c_1<\ldots<c_{p-1}=1$, we know $c_{n-2}=a$ and

$$c_k = f_l^k(0) = b\sum_{i=0}^{k}\left(\frac{1-b}{a}\right)^i \text{ for } k=1,\ldots,p-3. \tag{4.14}$$

Therefore, equation (4.13) simplifies to the following:

$$(c_1-c_0)h_1+\ldots+(c_{p-1}-c_{p-2})h_{p-1} = 1 \tag{4.15}$$

$$b\sum_{i=0}^{p-3}\left(\frac{1-b}{a}\right)^i h_{i+1}+(1-a)h_{p-1} = 1. \tag{4.16}$$

With this additional constraint, the system of equations has dimension $p\times(p-1)$.

$$\mathbf{S}\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{p-1}\end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1\end{pmatrix}, \tag{4.17}$$

defining the coefficient matrix $\mathbf{S}$ of equation (4.17) as the mostly banded matrix

$$\mathbf{S} = \begin{pmatrix} -1 & & & & \dots & 0 & (1-a) \\ \frac{a}{1-b} & -1 & & & & \vdots & (1-a) \\ & \frac{a}{1-b} & -1 & & & & (1-a) \\ & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \frac{a}{1-b} & -1 & (1-a) \\ 0 & \dots & & & & \frac{a}{1-b} & -a \\ b & b\left(\frac{1-b}{a}\right) & & \dots & & b\left(\frac{1-b}{a}\right)^{p-3} & (1-a) \end{pmatrix}. \tag{4.18}$$

The next step is to prove that this new system has rank p. Name the rows in the coefficient matrix $\mathbf{S}$ as R_1, R_2, ..., R_p. Using the definition of linear dependence, there must exist a set of constants that multiply the rows of the matrix so that they sum to zero,

$$R_1 k_1 + R_2 k_2 + \ldots + R_p k_p = \vec{0}. \tag{4.19}$$

Solving for these constants, we find

$$\begin{aligned} k_{p-i} &= k_{p-1}\left(\tfrac{a}{1-b}\right)^i + k_p\ b\ (i-1)\left(\tfrac{a}{1-b}\right)^{i-p+2} \text{ for } i = 2, \ldots, p-1 \\ k_{p-1} &= \frac{1-a}{a}\left(\sum_{i=1}^{p-2} k_i + 1\right). \end{aligned}$$

There is only one degree of freedom and the system has rank $p-1$. There is a unique solution to the system, if one exists.

The closed form for the constants h_i in the probability density function are

$$h_i = \frac{1-b}{1-b-ab(p-1)}\left(1-\left(\frac{a}{1-b}\right)^i\right) \text{ for } i = 1, \ldots, p-2 \tag{4.20}$$

$$h_{p-1} = \frac{1-a-b}{(1-a)(1-b-ab(p-1))}. \tag{4.21}$$

We show this calculation in Appendix A.

Example: The CRLL family illustrates the previous results. The Markov partition is formed by the period four orbit $(0, b, a, 1)$. See Figure 1 for a cobweb diagram of a specific period four orbit. In parameter space, the family occurs on the curve $a = f_L(b)$, or

$$b = \frac{1+a-\sqrt{1+2a-3a^2}}{2}. \tag{4.22}$$

Note that the other branch of the solution occurs outside the parameter domain, $(a,b) \in (0,1) \times [0,1]$. Equation (4.5) produces the system of equations

$$\begin{aligned} h_1 &= (1-a)\,h_3 \\ h_2 &= \left(\frac{a}{1-b}\right) h_1 + (1-a)\,h_3 \\ h_3 &= \left(\frac{a}{1-b}\right) h_2 + (1-a)\,h_3. \end{aligned} \tag{4.23}$$

To normalize h, set the area under the curve to one,

$$bh_1 + (a-b)h_2 + (1-a)h_3 = 1. \tag{4.24}$$

Hence, the invariant density function is

$$h(x) = \begin{cases} \frac{1-a-b}{1-b-3ab} & \text{if } 0 \le x < b \\ \frac{(1-b)^2-a^2}{(1-b)(1-b-3ab)} & \text{if } b \le x < a \\ \frac{1-a-b}{(1-a)(1-b-3ab)} & \text{if } a \le x \le 1 \end{cases}.$$

See Figure 1 for a comparison of this exact result to a histogram generated by a "typical" orbit.

Of particular use, the invariant measure can be used to exactly calculate the Lyapunov exponent for any set of parameters (a, b) along the curves described by equation (4.2) in D_0. WOLOG, assign $c_k = a$ and $k = p - 2$. Use equations (3.1), (4.20), and (4.21) to derive the following:

$$\begin{aligned} \Lambda_{a,b} &= \ln\left|\tfrac{1-b}{a}\right| \textstyle\sum_{i=1}^{k}(c_i - c_{i-1})h_i + \ln\left|\tfrac{1}{1-a}\right| \textstyle\sum_{i=k+1}^{p-1}(c_i - c_{i-1})h_i \\ &= \ln\left|\tfrac{1-b}{a}\right|(1-(1-a)h_{p-1}) + \ln\left|\tfrac{1}{1-a}\right|(1-a)h_{p-1} \\ &= \ln\left|\tfrac{1-b}{a}\right| + \ln\left|\tfrac{a}{(1-a)(1-b)}\right|(1-a)h_{p-1} \\ &= \ln\left|\tfrac{1-b}{a}\right| + \ln\left|\tfrac{a}{(1-a)(1-b)}\right|\left(\tfrac{1-a-b}{1-b-ab(p-1)}\right). \end{aligned} \tag{4.25}$$

Example: Continuing the period four CRLL example from above, we derive the Lyapunov exponent exactly. Set $p = 4$ and in equation (4.25),

$$\Lambda_{a,b} = \ln\left|\tfrac{1-b}{a}\right| + \ln\left|\tfrac{a}{(1-a)(1-b)}\right|\left(\tfrac{1-a-b}{1-b-3ab}\right). \tag{4.26}$$

In parameter space, the family occurs on the curve derived in equation (4.22). Therefore, set $b = (1 + a - \sqrt{1+2a-3a^2})/2$, which make makes the Lyapunov exponent a function of one parameter, a. See Figure 2 for a graph of equation (4.26) as a function of a. Because the Lyapunov exponent is positive for $0 < a < 1$, $F_{a,b}$ must be chaotic on this entire curve, not just in the region D_0 of equation (1.3).

5. Non-Markov Transformations

In this section, we prove that Markov maps are dense in D_0. Then, recalling results concerning weak limits of invariant measures of a sequence of transformations, we note that the Markov techniques can be applied, in a limiting sense, to describe statistical properties for all $F_{a,b}$ with $a, b \in D_0$. We begin by noting previous work on the map. The following is the proof that the map $f_{\lambda,\mu}$ studied in [4] is conjugate to the piecewise linear, interval map $F_{a,b}$ we are studying.

$$f_{\lambda,\mu}(x) = \begin{cases} 1 + \lambda x & \text{if } x \le 0 \\ 1 - \mu x & \text{if } x \ge 0. \end{cases}$$

with $\lambda \le 1$, $\mu > 1$, $0 < a < 1$, and $0 \le b \le 1$.

Using the conjugacy $\vartheta(x) = (x-a)/(1-a)$, show that $\vartheta^{-1} \circ f_{\lambda,\mu} \circ \vartheta(x) = F_{a,b}(x)$. Since $\vartheta(x)$ is a linear function of x, it is a homeomorphism and ϑ^{-1} is uniquely defined as $\vartheta^{-1}(x) = (1-a)x + a$.

$$\vartheta^{-1}(f(\vartheta(x))) = \begin{cases} (1 - a\lambda) + \lambda x & \text{if } x \le a \\ (1 + a\mu) - \mu x & \text{if } x \ge a. \end{cases} \tag{5.1}$$

Let $\lambda = (1-b)/a$ and $\mu = 1/(1-a)$. Then

$$1 - a\lambda = 1 - a\left(\frac{1-b}{a}\right) = b \tag{5.2}$$

$$1 + a\mu = 1 - \mu\left(\frac{1}{1-a}\right) = \frac{1}{1-a}. \tag{5.3}$$

Therefore, $\vartheta^{-1} \circ f_{\lambda,\mu} \circ \vartheta(x) = F_{a,b}(x)$ by the homeomorphism ϑ and the two maps $F_{a,b}$ and $f_{\lambda,\mu}$ are topologically conjugate.

Call $\mathcal{M}$ the class of sequences $\mathbf{M}$ which occur as kneading sequences of $F_{a,\frac{1-2a}{1-a}}$ for $0 < a \leq 1/2$, also known as the primary sequences. Misiurewicz and Visinescu proved the following theorems:

Theorem 5.1 ([4], Theorem A)**.** *If (a,b), $(a',b') \in D$ and $(a,b) > (a',b')$ then $K(a,b) > K(a',b')$.*

Theorem 5.2 ([4], Theorem B)**.** *If $(a,b) \in D$ then $K(a,b) \in \mathcal{M}$.*

Theorem 5.3 ([6], Intermediate Value Theorem for Kneading Sequences)**.** *If a one-parameter family G_t of continuous unimodal maps depends continuously on t and $h(G_t) > 0$ for all t then if $K(G_{t_0}) < K < K(G_{t_1})$ and $K \in \mathcal{M}$ then there exists t between t_0 and t_1 with $K(G_t) = K$.*

Using these previous results, we can prove the following,

Theorem 5.4. *$\forall (a,b) \in D_0$, one of the following is true:*

1. *$F_{a,b}$ is Markov.*
2. *$\exists (a^*,b^*) D_0$ such that F_{a^*,b^*} uniformly approximates $F_{a,b}$.*

Proof. If $F_{a,b}$ is Markov, we are done. Otherwise, choose F_{a_0,b_0} such that $K(F_{a_0,b_0})$ is nonperiodic. Given a small $\varepsilon > 0$, let $a_0 = a^*$ and $b_1 = b_0 + \varepsilon$. By Theorem 5.1, $K(F_{a^*,b_0}) < K(F_{a^*,b_1})$ and by Theorem 5.2, $K(F_{a^*,b_0})$, $K(F_{a^*,b_1}) \in \mathcal{M}$. WOLOG, we choose indices of b_0 and b_1 to create this ordering. Recall by Lemma 2.4, that periodic sequences are dense in Σ_2. Therefore, we may choose a sequence $M \in \mathcal{M}$ such that $K(F_{a^*,b_0}) < M < K(F_{a^*,b_1})$. Since $F_{a,b}$ does vary continuously with parameters a and b, then Theorem 5.3 implies an intermediate value b^* such that $b_0 < b^* < b_1$, and this intermediate map has the decimal kneading $K(F_{a^*,b^*}) = M$. Therefore, in any given neighborhood of a non-Markov map in D_0, there exists a Markov map M. ∎

Hence, we can construct a sequence, in D_0, of Markov maps that converges to any $F_{a,b}$ with $(a,b) \in D_0$. Considering our Theorem 5.4, and the following result [1], we conclude that any transformation $F_{a,b}$ with $(a,b) \in D_0$ is either a member of the Markov set which we constructed, and the invariant density function can be calculated directly as described earlier in this paper, or if $F_{a,b}$ is not in that set, then a sequence of uniformly convergent Markov transformations, $F_{a_i,b_i} \to F_{a,b}$ and $(a_i,b_i) \to (a,b)$, each have easily calculated invariant densities which converge to the invariant density of $F_{a,b}$.

Define Q be the set $\{c_0, c_1, \ldots, c_{p-1}\}$ and $\mathcal{P}$ be the partition of I into closed intervals with endpoints belonging to $Q : I_1 = [c_0, c_1], \ldots, I_{p-1} = [c_{p-2}, c_{p-1}]$.

Theorem 5.5 ([1], Theorem 10.3.2)**.** *Let $f : I \to I$ be a piecewise expanding transformation, and let $\{f_n\}_{n\geq 1}$ be a family of Markov transformations associated with f. Note $Q^{(0)} = Q$, and*

$$Q^{(k)} = \bigcup_{j=0}^{k} f^{-j}(Q^{(0)}), \ k = 1, 2, \ldots$$

Moreover, we assume that $f_n \to f$ uniformly on the set

$$I \setminus \bigcup_{k\geq 0} Q^{(k)}$$

and $f'_n \to f'$ in $\mathcal{L}^1$ as $n \to \infty$. Any f_n has an invariant density h_n and $\{h_n\}_{n\geq 1}$ is a precompact set in $\mathcal{L}^1$. We claim that any limit point of $\{h_n\}_{n\geq 1}$ is an invariant density of f.

6. Conclusion

For the two-parameter family of skew tent maps, we have shown that there is a dense set of parameters in the chaotic region for which the maps are Markov and we exactly find the probability density function for any of these maps. It is known, [1], that when a sequence of transformations has a uniform limit, and the corresponding sequence of invariant probability density functions has a weak limit, then the limit of the probability density functions is invariant under the limit of the transformations. We construct such a sequence entirely within the family of skew tent maps amongst the set of Markov transformations. Numerical evidence that this is possible can be seen in Figure 4. The graph represents an approximation of how the probability density function (over the interval $[0, 1]$) changes as a function of the parameter b, when $a = 0.9 - b$. The "jumps" or "steps" vary continuously with the parameter, and we have highlighted the example symbolic pattern CRL, CRLL, CRLLL, ..., for which the results were derived exactly in Section 4. We also presented an application of this work in the exact calculation of Lyapunov exponents.

7. Acknowledgements

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Appendix A. Probability density function calculation

From equation (4.6), we see the pattern

$$\begin{aligned}
h_i &= h_{p-1}(1-a)\sum_{j=0}^{i-1}\left(\frac{a}{1-b}\right)^j \text{ for } i=1,\dots,p-2 \\
&= h_{p-1}(1-a)\left(\frac{1-\left(\frac{a}{1-b}\right)^i}{1-\left(\frac{a}{1-b}\right)}\right) \\
&= h_{p-1}\frac{(1-a)(1-b)\left(1-\left(\frac{a}{1-b}\right)^i\right)}{(1-a-b)}.
\end{aligned} \tag{A.1}$$

Substitute this expression for h_i in equation 4.16.

$$\begin{aligned}
b\sum_{i=0}^{p-3}\left(\frac{1-b}{a}\right)^i h_{i+1} + (1-a)h_{p-1} &= 1 \\
b\, h_{p-1}\sum_{i=0}^{p-3}\frac{(1-a)(1-b)}{(1-a-b)}\left(1-\left(\frac{a}{1-b}\right)^{i+1}\right)\left(\frac{1-b}{a}\right)^i + (1-a)h_{p-1} &= 1 \\
h_{p-1}\frac{b(1-a)(1-b)}{(1-a-b)}\sum_{i=0}^{p-3}\left(\left(\frac{1-b}{a}\right)^i - \left(\frac{a}{1-b}\right)\right) + (1-a)h_{p-1} &= 1 \\
h_{p-1}\frac{(1-a)(1-b)}{(1-a-b)}\left(\sum_{i=0}^{p-3} b\left(\frac{1-b}{a}\right)^i - \sum_{i=0}^{p-3}\left(\frac{ab}{1-b}\right)\right) + (1-a)h_{p-1} &= 1 \\
h_{p-1}\frac{(1-a)(1-b)}{(1-a-b)}\left(a-\left(\frac{ab(p-2)}{1-b}\right)\right) + (1-a)h_{p-1} &= 1 \\
h_{p-1}\frac{a(1-a)(1-b-b(p-2))}{(1-a-b)} + (1-a)h_{p-1} &= 1.
\end{aligned} \tag{A.2}$$

Solving this equation for h_{p-1}, we find

$$\begin{aligned}
h_{p-1} &= \left(\frac{a(1-a)(1-b-b(p-2))}{(1-a-b)} + (1-a)\right)^{-1} \\
&= \frac{(1-a-b)}{(1-a)\left(a(1-b-b(p-2)) + (1-a-b)\right)} \\
&= \frac{(1-a-b)}{(1-a)\left(1-b-ab(p-1)\right)}.
\end{aligned} \tag{A.3}$$

Therefore, h_i can be expressed as

$$h_i = \frac{1-b}{1-b-ab(p-1)}\left(1-\left(\frac{a}{1-b}\right)^i\right) \text{ for } i=1,\dots,p-2. \tag{A.4}$$

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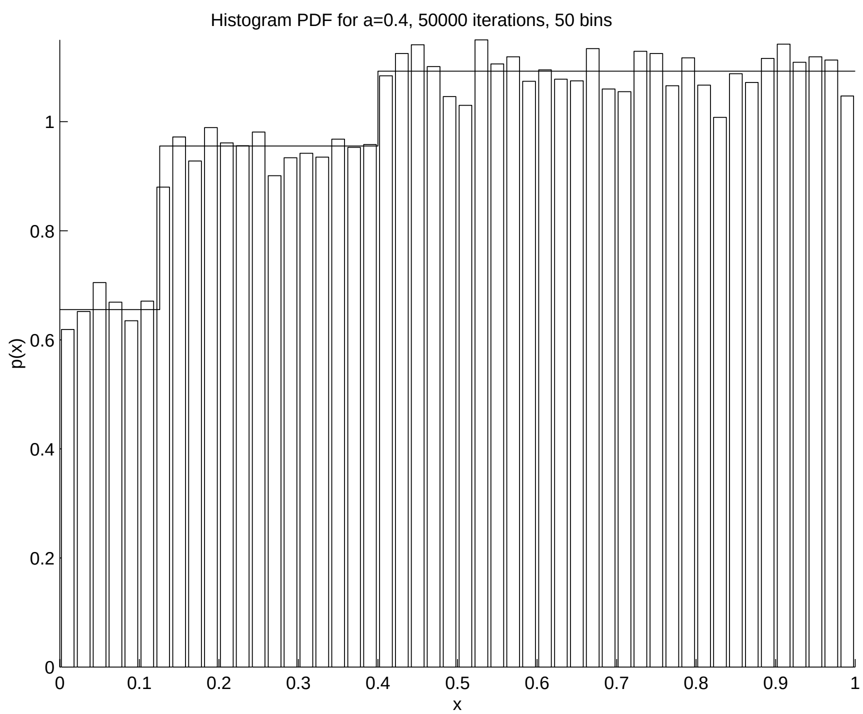

FIGURE 1. Numerical calculation of the PDF (histogram bars) compared to the exact solution (solid line). The calculation used 50,000 iterations and a 50 intervals.

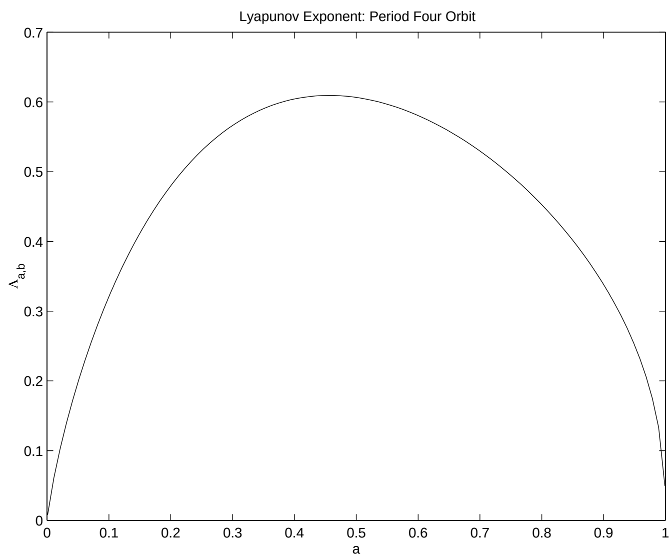

FIGURE 2. The Lyapunov Exponent as a function of the parameter a where $x = 0$ is a member of a period four orbit.

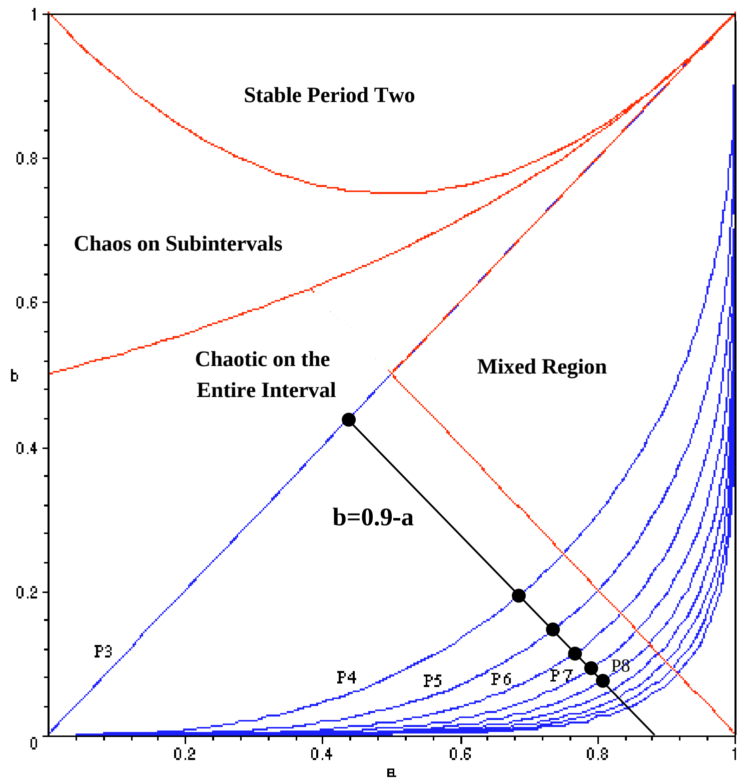

FIGURE 3. The line through parameter space associated with the numerical approximation of the probability density function shown in Figure 4, $a = 0.9 - b$. The points are the exact parameters for the period three through period eight Markov maps.

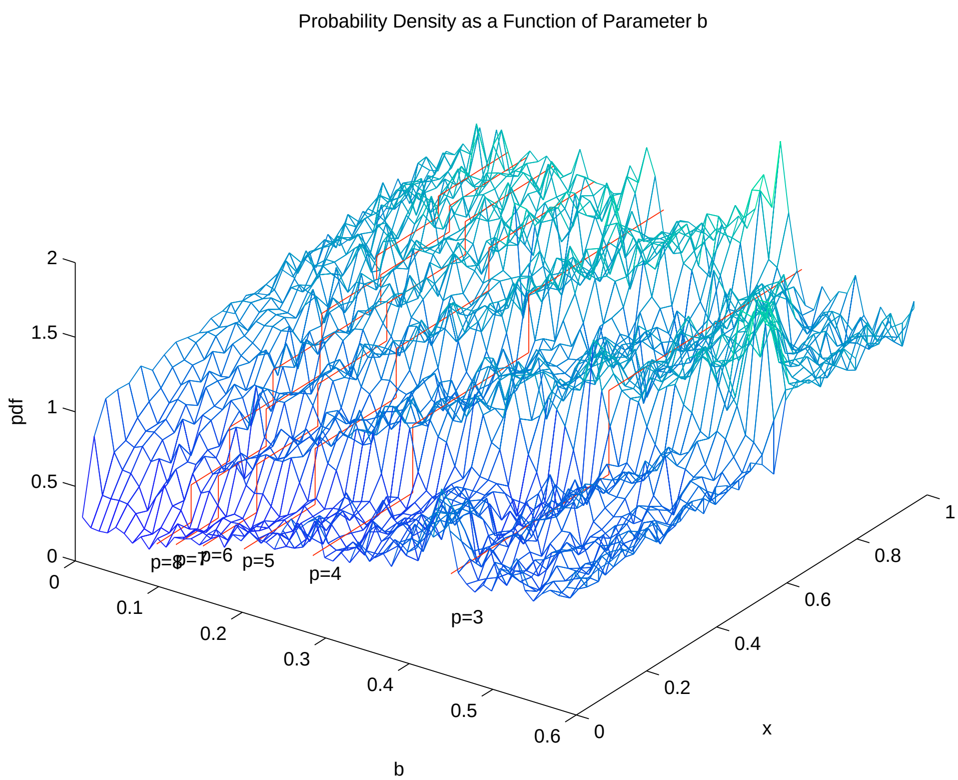

FIGURE 4. A numerical approximation of the probability density function as a function of the parameter b with $a = 0.9 - b$. The red lines are the exact solutions for the period three through period eight orbits.