REGULARITY OF COMMUTER FUNCTIONS FOR HOMEOMORPHIC DEFECT MEASURE IN DYNAMICAL SYSTEMS MODEL COMPARISON

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Abstract. In the field of dynamical system, conjugacy describes an equivalent relation between two dynamical systems. In our work, we are dealing with mostly conjugacy, which relates two dynamical systems that are not necessarily conjugate. We generate a function called "commuter" based on a fixed point iteration scheme. The resulting "commuter" is a nonhomeomorphic change of coordinates translating between two systems. And we can determine the amount of failing to be conjugacy, which we call homeomorphic defect, by studying the properties of commuters.

We consider the function space $L^p[0,1]$, with $1 \leq p < \infty$, and the norm is given by the standard L^p norm. We derive a contractive operator which will give a limit point from the commuting relationship even when applied to nonconjugate systems. We discuss the measurability of commuters. Specially, when studying behaviors of commuters between full symmetric tent map and short symmetric tent map, we show that the commuter is monotonely convergent to identity function as the height is going to 1. At last, we also give a computation error analysis for our computation method in producing commuters.

1. Introduction

In the field of dynamical systems, the concept conjugacy describes an equivalent relation between dynamical systems. More precisely, we have the following definition,

Definition 1.1. *Let X and Y be topological spaces, and let $g_1 : X \to X$ and $g_2 : Y \to Y$. The dynamical systems g_1 and g_2 are conjugate if there exists a homeomorphism $h : X \to Y$, such that*

$$g_1(x) = (h^{-1} \circ g_2 \circ h)(x) \tag{1}$$

for all $x \in X$.

Topologically, h is a function which is continuous, 1-to-1, onto, inverse continuous. If we rewrite (1.1) in a equivalent form

$$(h \circ g_1)(x) = (g_2 \circ h)(x) \tag{2}$$

We notice that (1.2) allow us to relax some conditions for the definition of conjugacy. We will introduce the concept of mostly conjugacy in the following definition to describe a function which satisfies (1.2), but not necessary a homeomorphism.

Definition 1.2. *Let X and Y be topological spaces, and let $g_1 : X \to X$ and $g_2 : Y \to Y$. If $f : X \to Y$ satisfies the commuting relationship (1.2), then we say f is a commuter relating the dynamical system g_1 and g_2.*

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Notice that this definition doesn't require the function f to be continuous, 1-to-1, onto or inverse continuous. Such commuters exit and computable. In paper[3], we build a scheme to construct commuter functions between the family of skew tent maps and the family of symmetric tent maps. When generating the sequence of the commuters in the scheme, we assume the commuters are in the space $B([0,\ 1],\ R)$, the set of all bounded functions from $[0,\ 1]$ to the real numbers, which is a Banach space, with the norm $||f|| = ||f||_\infty := sup|f(x)|$. It is the purpose of this paper to discuss a more general condition for the commuters function.

In section 2, we continue the example in paper[3], but consider the related functions in L^p space, with $1 \leqslant p < \infty$. In section 3, we will prove that, if a skew tent map converges to a full tent map, the commuter between this skew tent map and the full tent map converges to the identity function. In section 4, we show that using our scheme to generate functions are all measurable if the initial guess function is, and hence the limit of the sequence, the commuter, is also measurable. Together with the assumption that all the functions we are discussing are bounded, we should have the conclusion that the commuter is in L^p. In section 5, we sketch the computation work of presenting the commuter, and the error analysis for the technique that we are using.

2. A Contraction Mapping From the Commutative Relationship

In this section, we start with a simple but typical example, in which we see the exitance and uniqueness of such commuter function, and how to construct this commuter.

Consider the family of skew tent maps $S(x)$ defined on $[0, 1]$, with:
(1) $S(0) = 0,\ S(1) = 0$;
(2) Peak of tent occurs at $S(a) = b$ with $0 < a < 1$;
(3) To ensure the map is locally expanding, require $max(a, 1 - a) < b \leqslant 1$.
Define this family $S_{a,b}$

Consider the subset of maps that are symmetric about $x = 1/2$, denote the subfamily as $\tau,\ \tau \subset S,\ \tau_v := S_{1/2,v}$. First, the following lemma gives the existence of a conjugacy:

Lemma 2.1. *Let $S_{a,b}$ be a particular member of S. Then there exists a v_0 s.t. $S_{a,b}$ is conjugate to $T(v_0) \in \tau$. ($S \circ h(x) = h \circ T(x)$)*

Secondly, we demand that the commuter function maps monotone segments of the symmetric tent map to monotone segments of the full tent map. Since in $[0, 1/2]$, the symmetric tent map is $T(x) = 2v_0x$. From (1.2), we have

$$S \circ h(x) = h(2v_0x) \tag{3}$$

Notice that h maps the domain $[0, 1/2]$ of of $S(x)$ to the domain $[0, a]$ of $T(x)$. i.e. $h[0, 1/2] = [0, a]$. Since $S(x) = b/ax$, together with (2.3), we get,

$$\frac{b}{a}h(x) = h(2v_0x) \tag{4}$$

Similarly, in $(1/2, 1]$, we can get

$$\frac{b}{1-a}(1 - h(x)) = h(2v_0(1 - x)) \tag{5}$$

Therefore, the conjugacy function $h(x)$ must satisfy:

$$h(x) = \begin{cases} \frac{a}{b}h(2v_0x), & 0 \leqslant x \leqslant 1/2, \\ 1 - \frac{1-a}{b}h(2v_0(1-x)), & 1/2 < x \leqslant 1. \end{cases}$$

And, a conjugacy should map turning points to turning points, that is

$$h(1/2) = a \tag{7}$$

Then at $x = 1/2$, we have

$$h(1/2) = a = \frac{a}{b}h(v_0) \tag{8}$$

and

$$h(v_0) = b \tag{9}$$

Since the conjugate function $h(x)$ satisfied:

$$h(x) = \begin{cases} \frac{a}{b}h(2v_0x), & 0 \leqslant x \leqslant 1/2, \\ 1 - \frac{1-a}{b}h(2v_0(1-x)), & 1/2 < x \leqslant 1. \end{cases}$$

Now use this equation as a guide, we create an operator whose fixed point will satisfy (1.2).

Consider the space $B([0,1],R)$ with norm $\|f\| = \|f\|_{L^p} := (\int_{[0,1]}|f|^p dx)^{\frac{1}{p}},\ 1 \leqslant p < \infty$, which is a Banach space, complete. More precisely, we give the definition of the L^p space in the following.

Definition 2.2 (Definition of L^p Space)**.** *If E is a measurable subset of R and p satisfies $1 \leqslant p < \infty$, then $L^p(E)$ denotes the collection of measurable f for which $\int_E |f(x)|^p dx$ is finite, that is*

$$L^p(E) = \{f : \int_E |f(x)^p| dx < \infty\},\ 1 \leqslant p < \infty.$$

We shall write

$$\|f\|_{p,E} = (\int_E |f(x)|^p dx)^{\frac{1}{p}},\ 1 \leqslant p < \infty$$

thus, $L^p(E)$ is the class of measurable f for which $\|f\|_{p,E}$ is finite.

From the closed subset $\mathcal{F} \subset B([0,1],R)$

$$\mathcal{F} = \{f | f : [0,1] \to [0,1]\}$$

Then given (a,b) satisfying $max(a, 1-a) < b < 1$ defined a one-parameter family of operators.

$$\mathcal{M}_v : \mathcal{F} \to \mathcal{F} \text{ for } 1/2 < v \leqslant 1$$

$$\mathcal{M}_v f(x) := \begin{cases} \frac{a}{b}f(2vx), & 0 \leqslant x \leqslant 1/2, \\ 1 - \frac{1-a}{b}f(2v(1-x)), & 1/2 < x \leqslant 1. \end{cases}$$

Consider on a, b and v are required to ensure $\mathcal{F}$ is mapping into itself, also cause the operator to be a contraction.

Lemma 2.3. *$\mathcal{M}_v$ is a uniform contraction on $\mathcal{F}$, where the contraction is with respect to $\|\cdot\|_p$.*

Proof. Define $\lambda = max(\frac{a}{b}, \frac{1-a}{b})$, then $0 \leqslant \lambda < 1$,

$$\|\mathcal{M}_v f_1 - \mathcal{M}_v f_2\|_{p,[0,1]} = (\int_{[0,1]} |\mathcal{M}_v f_1 - \mathcal{M}_v f_2|^p dx)^{\frac{1}{p}}$$

For $0 \leqslant x < 1/2$

$$\begin{aligned}
&(\int_{[0,1/2]} |\mathcal{M}_v f_1 - \mathcal{M}_v f_2|^p dx)^{\frac{1}{p}} \\
&= (\int_{[0,1/2]} |\frac{a}{b}(f_1(2vx) - f_2(2vx)|^p dx)^{\frac{1}{p}} \\
&= \frac{a}{b}(\int_{[0,1]} |f_1(y) - f_2(y)|^p d\frac{y}{2v})^{\frac{1}{p}} \\
&\leqslant \frac{a}{(2v)^{\frac{1}{p}} b}(\int_{[0,1]} |f_1(y) - f_2(y)|^p dy)^{\frac{1}{p}} \\
&\leqslant \frac{\lambda}{(2v)^{\frac{1}{p}}} \|f_1 - f_2\|_p
\end{aligned}$$

Similarly, for $1/2 \leqslant x < 1$,

$$\begin{aligned}
&(\int_{[1/2,1]} |\mathcal{M}_v f_1 - \mathcal{M}_v f_2|^p dx)^{\frac{1}{p}} \\
&= (\int_{[1/2,1]} |\frac{1-a}{b}(f_1(2vx) - f_2(2vx)|^p dx)^{\frac{1}{p}} \\
&\leqslant \frac{1-a}{b}(\int_{[0,1]} |f_1(y) - f_2(y)|^p d\frac{y}{2v})^{\frac{1}{p}} \\
&\leqslant \frac{1-a}{(2v)^{\frac{1}{p}} b}(\int_{[0,1]} |f_1(y) - f_2(y)|^p dy)^{\frac{1}{p}} \\
&\leqslant \frac{\lambda}{(2v)^{\frac{1}{p}}} \|f_1 - f_2\|_p
\end{aligned}$$

Thus, $\|\mathcal{M}_v f_1 - \mathcal{M}_v f_2\|_{p,[0,1]} \leqslant \lambda \|f_1 - f_2\|_{p,[0,1]}$.

□

So $\mathcal{M}$ is a contraction , with contraction constrain constant λ. Because λ not depends on v, the constant is uniform.

Lemma 2.4. *There is a unique $f_v \in \mathcal{F}$, so that*

$$\mathcal{M}_v f_v = f_v$$

Moreover, for an arbitrary $f_0 \in \mathcal{F}$, if we define the sequence of functions

$$f_{n+1} = \mathcal{M}_v f_n$$

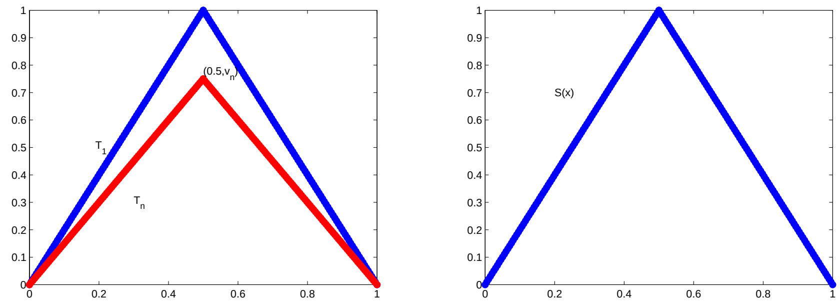
this sequence will converge to f_v

$$f_v := \lim_{n\to\infty} f_n$$

3. Convergence Of Commuter Function

In this section, we continuous the example in the last section, but consider a more theoretical issue.

In Figure 1, we know that a full tent map is self-conjugate because the identity function is a homeomorphism. But a symmetric tent map with lower height might not be conjugate to the full tent map. We are going to check if we shift the height of the symmetric tent map, what is going to happen to the commuter function. Would the commuter converges to the identity function? Or even more, monotone converges?



(a) Shift Tent Maps (b) Full Tent Map

FIGURE 1. (Color) Shift tent maps.(Left) A tent map T_n with peak at $(\frac{1}{2}, 1)$(blue), and a tent map T_n with peak at $(\frac{1}{2}, V_n)$(red). (Right) A full tent map S. Here $T_n = S$, so the commuter between them is the identity function I. As V_n going to 1, the commuter function between T_n and S is going to the identity function I under the norm $\|\cdot\|_{p,[0,1]},\ 1 \leqslant p < \infty$

Theorem 3.1. *If V_n is a sequence going to 1, then the commuter from the symmetric tent map T_n with peak V_n to the full symmetric tent map S, is convergent to the identity function $I(x)$ under the norm $\|\cdot\|_{p,[0,1]}$*

Proof. From section 2, it is easy to see the commuter function between T_n and S satisfies the following equations

$$h_n(x) = \begin{cases} \dfrac{1}{2}h_n(2v_n x), & 0 \leqslant x \leqslant \frac{1}{2}, \\ 1 - \dfrac{1}{2}h_n(2v_n(1-x)), & \frac{1}{2} < x \leqslant 1. \end{cases}$$

Note: Just take $a = \frac{1}{2},\ b = 1$ at the original equations.

Also notice that h_n maps monotone segments of the graph of T_n onto monotone segments of S. i.e. h_n maps segment on the interval $[0,\ \frac{1}{2}]$ of T_n onto the segment on the interval $[0,\ \frac{1}{2}]$ of S. Also notice that the inverse function of S exists on each of these intervals.

First consider $0 \leqslant x \leqslant \frac{1}{2}$,

$$\begin{aligned} \|h_n - I\|_{p,[0,\frac{1}{2}]} &= \|S^{-1} \circ h_n \circ T_n - S^{-1} \circ I \circ S\|_{p,[0,\frac{1}{2}]} \\ &\leqslant L\|h_n \circ T_n - I \circ S\|_{p,[0,\frac{1}{2}]} \end{aligned}$$

Since in this simple example, we are dealing with the system $S(x) = 2x$. Thus it is easy to see that the lipshitz for $S^{-1}(x)$ is $\frac{1}{2}$. However, we will point out the fact in the next part that for more general systems beside $S(x) = 2x$, the above idea of proof with still work.

Thus,

$$\begin{aligned}
\|h_n - I\|_{p,[0,\frac{1}{2}]} &\leqslant \frac{1}{2}\|h_n \circ T_n - I \circ S\|_{p,[0,\frac{1}{2}]} \\
&\leqslant \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[0,\frac{1}{2}]} + \frac{1}{2}\|h_n \circ S - I \circ S\|_{p,[0,\frac{1}{2}]} \\
&= \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[0,\frac{1}{2}]} + \frac{1}{2}(\int_0^{\frac{1}{2}} |h_n(2x) - 2x|^p dx)^{\frac{1}{p}}
\end{aligned}$$

Let $2x = u$, we can substitute $x = \frac{u}{2}$ on the above equation because of its absolute continuity.[3]
So,

$$\begin{aligned}
\|h_n - I\|_{p,[0,\frac{1}{2}]} &\leqslant \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[0,\frac{1}{2}]} + \frac{1}{2}(\int_0^{\frac{1}{2}} |h_n(2x) - 2x|^p dx)^{\frac{1}{p}} \\
&= \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[0,\frac{1}{2}]} + \frac{1}{2}(\int_0^{1} |h_n(u) - u|^p d\frac{u}{2})^{\frac{1}{p}} \\
&= \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[0,\frac{1}{2}]} + \frac{1}{2^{1+\frac{1}{p}}}\|h_n - I\|_{p,[0,1]}
\end{aligned} \tag{13}$$

On the other hand, for $\frac{1}{2} \leqslant x \leqslant 1$,

$$\begin{aligned}
\|h_n - I\|_{p,[\frac{1}{2},1]} &\leqslant L\|h_n \circ T_n - I \circ S\|_{p,[\frac{1}{2},1]} \\
&\leqslant L\|h_n \circ T_n - h_n \circ S\|_{p,[\frac{1}{2},1]} + L\|h_n \circ S - I \circ S\|_{p,[\frac{1}{2},1]} \\
&= \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[\frac{1}{2},1]} + \frac{1}{2}(\int_{\frac{1}{2}}^{1} |h_n(2(1-x)) - 2(1-x)|^p dx)^{\frac{1}{p}}
\end{aligned}$$

Again, let $2(1-x) = u$, we can substitute $x = 1 - \frac{u}{2}$ on the above equation because of its absolute continuity.[3]
So,

$$\begin{aligned}
\|h_n - I\|_{p,[\frac{1}{2},1]} &\leqslant \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[\frac{1}{2},1]} + \frac{1}{2}(\int_{\frac{1}{2}}^{1} |h_n(2(1-x)) - 2(1-x)|^p dx)^{\frac{1}{p}} \\
&= \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[\frac{1}{2},1]} + \frac{1}{2}(\int_{1}^{0} |h_n(u) - u|^p d(1-\frac{u}{2}))^{\frac{1}{p}} \\
&= \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[\frac{1}{2},1]} + \frac{1}{2}(\int_{0}^{1} |h_n(u) - u|^p \frac{1}{2} du)^{\frac{1}{p}} \\
&= \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[\frac{1}{2},1]} + \frac{1}{2^{1+\frac{1}{p}}}\|h_n - I\|_{p,[0,1]}
\end{aligned} \tag{14}$$

Since $1 \leqslant p < \infty$, by raising p power on (1)+(2), we can get

$$\begin{aligned}
\|h_n - I\|^p_{p,[0,\frac{1}{2}]} + \|h_n - I\|^p_{p,[\frac{1}{2},1]} &\leqslant (\|h_n - I\|_{p,[0,\frac{1}{2}]} + \|h_n - I\|_{p,[\frac{1}{2},1]})^p \\
&\leqslant (\frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[0,\frac{1}{2}]} + \frac{1}{2^{1+\frac{1}{p}}}\|h_n - I\|_{p,[0,1]} \\
&\quad + \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[\frac{1}{2},1]} + \frac{1}{2^{1+\frac{1}{p}}}\|h_n - I\|_{p,[0,1]})^p
\end{aligned}$$

So,

$$\|h_n - I\|_{p,[0,1]} \leqslant \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[0,\frac{1}{2}]} + \frac{1}{2}\|h_n \circ T_n - h_n \circ S\|_{p,[\frac{1}{2},1]} + \frac{1}{2^{\frac{1}{p}}}\|h_n - I\|_{p,[0,1]}$$

which implies

$$\|h_n - I\|_{p,[0,1]} \leqslant \frac{1}{2(1-\frac{1}{2^{\frac{1}{p}}})}\|h_n \circ T_n - h_n \circ S\|_{p,[0,\frac{1}{2}]} + \frac{1}{2(1-\frac{1}{2^{\frac{1}{p}}})}\|h_n \circ T_n - h_n \circ S\|_{p,[\frac{1}{2},1]}$$

By the continuity in L^p[2], since

$$\|T_n - S\|_{p,[0,\frac{1}{2}]} \to 0,\ \|T_n - S\|_{p,[\frac{1}{2},1]} \to 0\ (V_n \to 1)$$

we have

$$\|h_n \circ T_n - h_n \circ S\|_{p,[0,\frac{1}{2}]} \to 0,\ \|h_n \circ T_n - h_n \circ S\|_{p,[\frac{1}{2},1]} \to 0$$

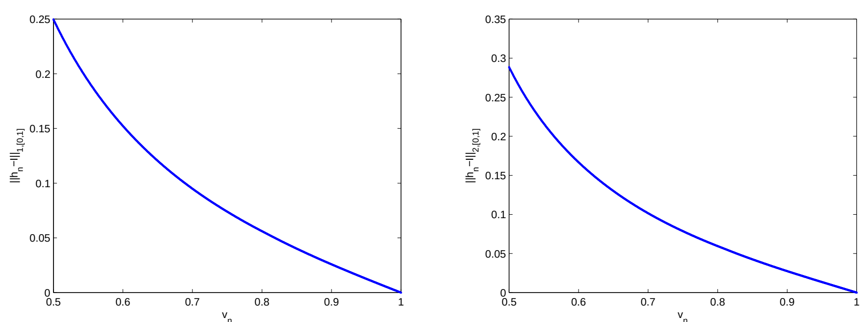
Thus

$$\|h_n - I\|_{p,[0,1]} \to 0\ as\ V_n \to 1,\ 1 \leqslant p < \infty.$$

□

The above theorem says that as $v_n \to 1,\ \|h_n - I\|_{p,[0,1]} \to 0$. In fact, the convergence under this norm is strictly monotone. First let's see what we can tell from Figure 2.

As we increasing the values of v_n, we calculate the difference between the symmetric tent map with peak v_n and the full symmetric tent map under 1-norm and 2-norm. We can observe from Figure 2 that the values $\|h_n - I\|_{1,[0,1]}$ and $\|h_n - I\|_{2,[0,1]}$ are decreasing monotone as v_n is increasing. But what we have claimed is only based on computation results and observation, it is natural to ask whether this property of monotone convergence of the commuters can be proven theoretically, and more generally. That is, if $v_n \to 1,\ \|h_n - I\|_{p,[0,1]} \xrightarrow{monotone} 0$.

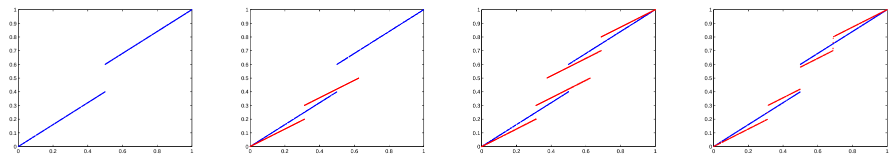


(a) Convergence Analysis with 1-norm (b) Convergence Analysis with 2-norm

FIGURE 2. As $v_n \to 1$, the difference between h_n and I under the norm $\|\cdot\|_{p,[0,1]}$ is going to 0 strictly monotone.

Before giving the answer, let's go back to the construction of the commuter functions and see whether it will give us some hits to prove it.

From the procedure of generating the commuter function between the symmetric tent map and the full symmetric tent map, we start with the identity function at the first iteration. Suppose we have the commuter function $h(x)$ at the n^{th} step. At the next step, we make a copy of $h(x)$, shrunk it by $1/2$ in the vertical and by $2v$ in the horizontal (Figure 3(b)). Take a second copy, scaled the same horizontally, and vertically scaled by $1/2$. Rotate this copy by 180 degrees and place it in the upper right portion of the unit square (Figure 3(c)). Then truncate the left copy to the interval $[0, 1/2)$ and the right copy to $[1/2, 1]$. The result (Figure 2(d)) return the commuter function $h(x)$ at step $n + 1^{th}$



(a) Commuter at 2^{nd} Iteration (b) Step 1 (c) Step 2 (d) Commuter at 3^{rd} Iteration

FIGURE 3. Construction Procedure Of Commuter Function

In Figure 4, the red curve is the commuter with the peak value higher than the blue ones. And the green line is the identity function. We can see that on $[0, 0.5]$, every point of the red one is greater than the green one. While on $[0.5, 1]$, every point of the red one is less than the green one. This makes the difference between the identity function and the 'red' commuter, i.e.$|I - h_{red}|$, is less than that between the identity function and the 'green' commuter i.e.$|I - h_{blue}|$, which also implies $||h_{red} - I||_{p,[0,1]} < ||h_{blue} - I||_{p,[0,1]},\ 1 \leq p < \infty$.

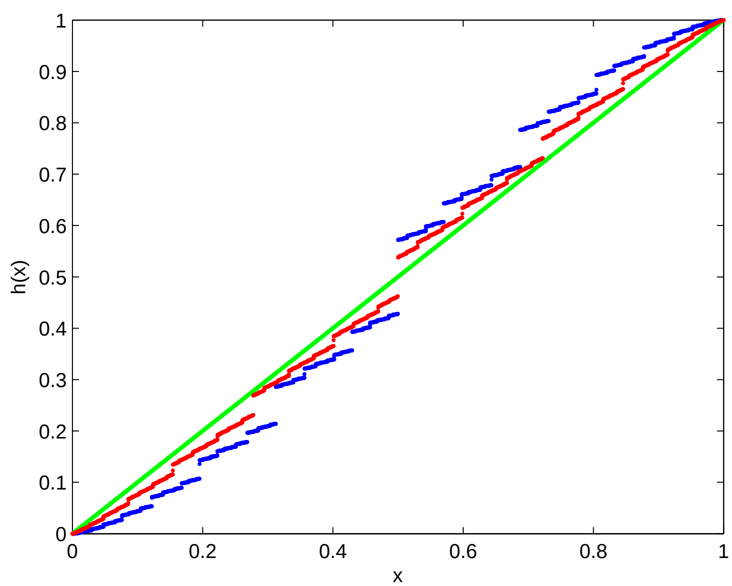


FIGURE 4. Analysis for the commuter function: (Green) The identity function. (Red) The commuter between the full tent map and the tent map with peak at $(\frac{1}{2}, 0.9)$. (Blue) The commuter between the full tent map and the tent map with peak at $(\frac{1}{2}, 0.8)$. Notice that on $[0, 0.5]$, every point of the red one is greater than the green one. While on $[0.5, 1]$, every point of the red one is less than the green one.

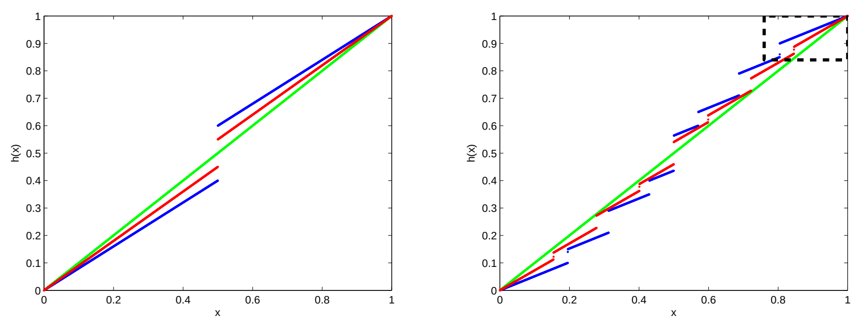
More precisely, if we can prove that given two symmetric tent maps, one is higher than the other, and they have the above property, then the monotone convergence follows.

Theorem 3.2. *If v_n is a sequence going to 1, then the commuter from the symmetric tent map T_n with peak v_n to the full symmetric tent map S, is monotone convergent to the identity function $I(x)$ under the norm* $\|\cdot\|_{p,[0,1]}$

Proof. Without loss of generality, we let v_{red} be the peak of the higher symmetric tent map, while v_{blue} be peak of the lower symmetric tent map. We are going to show that on the interval $[0, 1/2]$, every point of the commuter from the red symmetric tent map to the full tent map is greater than the blue one. on the interval $[1/2, 1]$, every point of the commuter from the blue symmetric tent map to the full tent map is greater than the red one. Here we just give the prove for the 1-norm. For arbitrary p-norm, it is also true by the fact of the embedding theorem.

We prove it by induction.

For the first iteration, we start with the identity function, see the Green line in Figure 5(a). Then we make a copy of $h(x)$, shrunk it by $1/2$ in the vertical and by $\frac{1}{2v_{red}}$, $\frac{1}{2v_{blue}}$ respectively in the horizontal. Take a second copy, scaled the same horizontally, and vertically scaled by $1/2$. Rotate this copy by 180 degrees and place it in the upper right portion of the unit square. Then truncate the left copy to the interval $[0, 1/2)$ and the right copy to $[1/2, 1]$. The result return the commuter function $h_{red}(x)$ and $h_{blue}(x)$ after the first iteration.



(a) 1^{th} Iteration (b) n^{th} iteration

FIGURE 5. Commuter Functions With Different Peak Values

We can see that the every red point in the interval $[0, 0.5]$ is greater than the blue one. Every blue point in the interval $[0.5, 1]$ is greater than the red one. That is true because the red one shrunk by $\frac{1}{2v_{red}}$, which is more than what the blue one shrunk. See Figure 5(a).

Now we suppose what we need to prove is true in the n^{th} iteration. That is, every red point in the interval $[0, 0.5]$ is greater than the blue one. Every blue point in the interval $[0.5, 1]$ is greater than the red one. At the next step, we still construct the commuter just like the procedure in the first iteration above. Without loss of generality and for simplification, we assume that the commuters look like Figure 5(b). In fact this picture is the commuter after three iterations. In order that at the $n + 1^{th}$ iteration, every red point is greater than the blue one in the interval $[0, 0.5]$, we just need to figure out in the most upper right segment, whether the red points shrunk more enough than the blue ones, so that every red point in this segment runs above the blue one. If so, then every point of the whole red commuter will runs above the blue commuter, which is exactly what we want to happen. This can be tell easier if see Figure 5(b). Since the blue line segment moves $1/2(1/v_{blue})$ horizontally, the red line segment moves $1/2(1/v_{red})$ horizontally, the blue one moves $1/2(1/v_{blue} - 1/v_{red})$ with respect to the red one. For n sufficiently large, the commuter is going to converge to the final commuter function, the slopes of the blue line segment and the red line segment are going to be 0. At the same time the length of them are going to be 0. So the left end points of the red and blue line segments will be closer and closer as $n \to \infty$, until their distance reaches 0, which is absolutely less than $1/2(1/v_{blue} - 1/v_{red})$. Thus the conclusion is true in $[0, 0.5]$. For $[0.5, 1]$, the argument is similar because of symmetry.

As a result, at the $n+1^{th}$ iteration, the conclusion is also true. This finishes the proof. □

Although the above discussion dealing with very simple cases, the idea of proof can be extend to be more general.

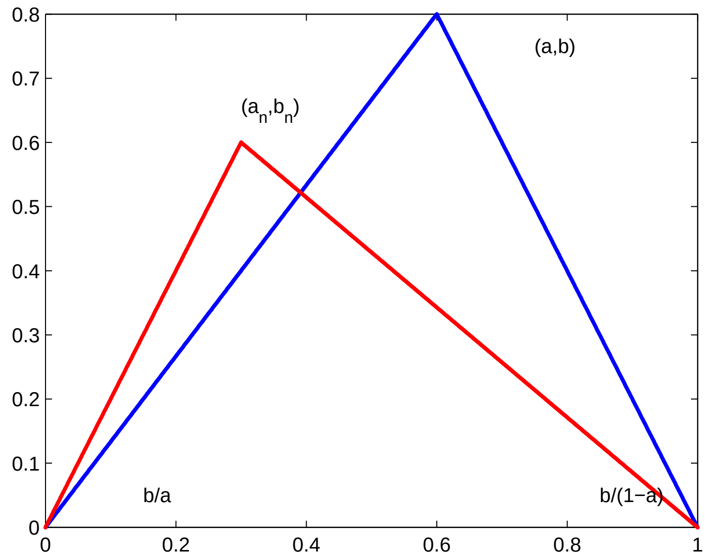


FIGURE 6. Convergence Analysis For Skew Tent Maps

For example, if we start with a pair of skew maps with peaks (a_n, b_n) and (a, b) respectively, see Figure 6. Let h_n be the commuter between these two systems. As the sequence $\{(a_n, b_n)\}$ goes to (a, b), it can be also showed that $\|h_{a_n,b_n} - I\|_{p,[0,1]} \to 0$. More precisely, we have the following theorem.

Theorem 3.3. *If (a_n, b_n) is a pair of sequence going to $(a, 1)$, then the commuter from the skew tent map T_n with peak (a_n, b_n) to the skew tent map T with peak $(a, 1)$, say h_n, is convergent to the identity function $I(x)$ under the norm $\|\cdot\|_{p,[0,1]}$,* $1 \leq p < \infty$

Proof. From the commutative diagram in section 2, we can have the commuter function between T_n and T satisfies the following equations

$$h_n(x) = \begin{cases} a h_n(\frac{b_n}{a_n} x), & 0 \leqslant x \leqslant a, \\ 1 - (1-a) h_n(\frac{b_n}{a_n}(1-x)), & a < x \leqslant 1. \end{cases}$$

Also notice that h_n maps monotone segments of the graph of T_{a_n,b_n} onto monotone segments of T. i.e. h_n maps segment on the interval $[0, a_n]$ of T_{a_n,b_n} onto the segment on the interval $[0, a]$ of T. Also notice that the inverse function of T exists on each of these intervals.

First consider $0 \leqslant x \leqslant a$,

$$\begin{aligned} \|h_n - I\|_{p,[0,a]} &= \|T^{-1} \circ h_n \circ T_{a_n,b_n} - T^{-1} \circ I \circ T\|_{p,[0,a]} \\ &\leqslant L \|h_n \circ T_{a_n,b_n} - I \circ T\|_{p,[0,a]} \end{aligned}$$

Here, we are dealing with the system $T(x) = 1/ax$. Thus it is easy to see that the lipshitz for $T^{-1}(x)$ is a.

Thus,

$$\begin{aligned}||h_n - I||_{p,[0,a]} &\leqslant a||h_n \circ T_{a_n,b_n} - I \circ T||_{p,[0,a]} \\ &\leqslant a||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[0,a]} + a||h_n \circ T - I \circ T||_{p,[0,a]} \\ &= a||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[0,a]} + a\left(\int_0^a |h_n(\frac{1}{a}x) - \frac{1}{a}x|^p dx\right)^{\frac{1}{p}}\end{aligned}$$

Let $\frac{1}{a}x = u$, we can substitute $x = au$ on the above equation because of its absolute continuity.[3]

So,

$$\begin{aligned}||h_n - I||_{p,[0,a]} &\leqslant a||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[0,a]} + a\left(\int_0^a |h_n(\frac{1}{a}x) - \frac{1}{a}x|^p dx\right)^{\frac{1}{p}} \\ &= a||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[0,a]} + a\left(\int_0^1 |h_n(u) - u|^p dau\right)^{\frac{1}{p}} \qquad (16) \\ &= a||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[0,a]} + a^{1+\frac{1}{p}}||h_n - I||_{p,[0,1]}\end{aligned}$$

On the other hand, for $a \leqslant x \leqslant 1$, we are dealing with the system $T(x) = 1/(1-a)(1-x)$. Thus it is easy to see that the lipshitz for $T^{-1}(x)$ is $(1-a)$.

$$\begin{aligned}||h_n - I||_{p,[a,1]} &\leqslant L||h_n \circ T_{a_n,b_n} - I \circ T||_{p,[a,1]} \\ &\leqslant L||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[a,1]} + L||h_n \circ T - I \circ T||_{p,[a,1]} \\ &= (1-a)||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[a,1]} + (1-a)\left(\int_a^1 |h_n(\frac{1}{1-a}(1-x)) - \frac{1}{1-a}(1-x)|^p dx\right)^{\frac{1}{p}}\end{aligned}$$

Again, let $\frac{1}{1-a}(1-x) = u$, we can substitute $x = 1-(1-a)u$ on the above equation because of its absolute continuity.[3]

So,

(17)

$$\begin{aligned}||h_n - I||_{p,[a,1]} &\leqslant (1-a)||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[a,1]} + (1-a)\left(\int_a^1 |h_n(\frac{1}{1-a}(1-x)) - \frac{1}{1-a}(1-x)|^p dx\right)^{\frac{1}{p}} \\ &= (1-a)||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[a,1]} + (1-a)\left(\int_1^0 |h_n(u) - u|^p d(1-(1-a)u)\right)^{\frac{1}{p}} \\ &= (1-a)||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[a,1]} + (1-a)\left(\int_0^1 |h_n(u) - u|^p (1-a)du\right)^{\frac{1}{p}} \\ &= (1-a)||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[a,1]} + (1-a)^{1+\frac{1}{p}}||h_n - I||_{p,[0,1]}\end{aligned}$$

Since $1 \leqslant p < \infty$, by raising p power on (16)+(17), we can get

$$\begin{aligned}||h_n - I||^p_{p,[0,a]} + ||h_n - I||^p_{p,[a,1]} &\leqslant (||h_n - I||_{p,[0,a]} + ||h_n - I||_{p,[a,1]})^p \\ &\leqslant (a||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[0,a]} + a^{1+\frac{1}{p}}||h_n - I||_{p,[0,1]} \\ &\quad + (1-a)||h_n \circ T_{a_n,b_n} - h_n \circ S||_{p,[a,1]} + (1-a)^{1+\frac{1}{p}}||h_n - I||_{p,[0,1]})^p\end{aligned}$$

So,

$$\begin{aligned}||h_n - I||_{p,[0,1]} &\leqslant a||h_n \circ T_{a_n,b_n} - h_n \circ T||_{p,[0,a]} + a^{1+\frac{1}{p}}||h_n - I||_{p,[0,1]} \\ &\quad + (1-a)||h_n \circ T_{a_n,b_n} - h_n \circ S||_{p,[a,1]} + (1-a)^{1+\frac{1}{p}}||h_n - I||_{p,[0,1]}\end{aligned}$$

As we assume that $\max\{a, 1-a\} < 1$, so $a < 1$ and $(1-a) < 1$. We also notice that $a^{1+\frac{1}{p}} < a$ and $(1-a)^{1+\frac{1}{p}} < (1-a)$. So here we have $a^{1+\frac{1}{p}} + (1-a)^{1+\frac{1}{p}} < a + (1-a) < 1$. Rearrange the above equation, we get

$$\begin{aligned}\|h_n - I\|_{p,[0,1]} \leqslant & \frac{a}{(1 - a^{1+\frac{1}{p}} - (1-a)^{1+\frac{1}{p}})} \|h_n \circ T_{a_n,b_n} - h_n \circ T\|_{p,[0,a]} \\ & + \frac{1-a}{(1 - a^{1+\frac{1}{p}} - (1-a)^{1+\frac{1}{p}})} \|h_n \circ T_{a_n,b_n} - h_n \circ T\|_{p,[a,1]}\end{aligned}$$

By the continuity in L^p[2], since

$$\|T_{a_n,b_n} - S\|_{p,[0,a]} \to 0,\ \|T_{a_n,b_n} - S\|_{p,[a,1]} \to 0\ ((a_n, b_n) \to (a, 1))$$

we have

$$\|h_n \circ T_{a_n,b_n} - h_n \circ T\|_{p,[0,a]} \to 0,\ \|h_n \circ T_{a_n,b_n} - h_n \circ T\|_{p,[a,1]} \to 0$$

Thus

$$\|h_n - I\|_{p,[0,1]} \to 0\ as\ (a_n, b_n) \to (a, 1),\ 1 \leqslant p < \infty.$$

□

4. The Measurability Of Commuter Functions

Theorem 4.1. *For the procedure of producing the commuter functions sequence f_n. If f_1, the initial guess commuter, is measurable, i.e. the identity function, the sequence f_n is measurable.*

Remark 4.2. f is measurable means $\{x \in E : f(x) > a\}$ is measurable for every finite a, E is measurable. Since $f_{n+1}[I_{X_i}] = g_{2i}^{-1} \circ f_n \circ g_1[I_{X_i}]$, g_{2i}^{-1} is continuous on Y. $f_n : X \to Y$. Suppose the range of $g_1[I_{X_i}]$ is a measurable set.

Lemma 4.3. *If φ is continuous, f is finite a.e. and measurable on E, then $\varphi \circ f$ is measurable.*

Remark 4.4. Please be noticed that the product of two measurable functions may not be measurable. That's the reason that I force the assumption of the measurability of the range of $g_1[I_{X_i}]$. (More details are in 'Proof of Statement 1') In fact, this assumption would not be too strong since the dynamical systems g_1, g_2 in our project are 'regular' in some sense.

Proof of Theorem 11. From the relation that

$$f_2[I_{X_i}] = g_{2i}^{-1} \circ f_1 \circ g_1[I_{X_i}],\ g_{2i}^{-1}$$

Since f_1 is measurable on the range of $g_1[I_{X_i}]$ as we suppose, i.e. most of the time we will start with $f_1 = I$, identity function.

So f_2 is measurable, so is $f_3 ... f_n$, by 4.3. □

Lemma 4.5. *If $f_n$is a sequence of measurable functions, then* $\sup_k f_k(x)$ *and* $\inf_k f_k(x)$ *are measurable. Here* $\{x : \sup_k f_k(x) > a\} = \cup_k \{x : f_k(x) > a\}$

Proof. [1]Since $\inf_k f_k = -\sup_k(-f_k)$, it is enough to prove the result for $\sup_k f_k$. This follow from the fact that $\{\sup_k f_k > a\} = \cup_k \{f_k > a\}$. □

Theorem 4.6. *From **lemma 2**, sequence f_n converges to f. We claim that f is also measurable, and hence f is in L^p because we assume the commuter function f is bounded.*

Proof. [1]Since

$$\lim_{k\to\infty}\sup f_k = \inf_j\{\sup_{k\geqslant j} f_k\}$$
$$\lim_{k\to\infty}\inf f_k = \sup_j\{\inf_{k\geqslant j} f_k\}$$

Then this statement follows from 4.5. Since $\lim_{k\to\infty} f_k$ exists, so it equals to $\lim_{k\to\infty}\sup f_k$ and $\lim_{k\to\infty}\inf f_k$, and hence measurable. □

5. An Improvement On Piecewise Interpolation

For the computation error analysis, we use the method in the following diagram:

$$f \xrightarrow{Blur} f_\epsilon \xrightarrow{PiecewiseInterpolating} \hat{f}$$

Where f is the original commuter, in L^p space, $1 \leqslant p < \infty$. While $f_\epsilon = f * K_\epsilon = \int f(x-t)K_\epsilon(t)dt$ and $\hat{f}$ is the piecewise interpolation to f_ϵ.

Remark 5.1. Define $K_\epsilon(x) = \epsilon^{-n}K(\frac{x}{\epsilon}) = \epsilon^{-n}K(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}, ..., \frac{x_n}{\epsilon})$ with $K \in L^1(R^n)$, $\epsilon > 0$, then it follows,

$$(i)\ \int_{R^n} K_\epsilon = \int_{R^n} K = 1;$$
$$(ii)\ \int_{|x|>\delta} |K_\epsilon| \to 0\ as\ \epsilon \to 0,\ for\ any\ fixed\ \delta > 0$$

Theorem 5.2. *If $f \in L^p(R^n)$, $1 \leqslant p < \infty$, then $\forall\eta > 0$, $\exists\delta > 0$, s.t.*

$$\begin{aligned} \|f - \hat{f}\|_p &\leqslant \|f - f_\epsilon\|_p + \|f_\epsilon - \hat{f}\|_p \\ &\leqslant \|K\|_1^{\frac{p}{p'}} \cdot [\eta\|K\|_1 + (2\|f\|_p)^p \int_{|t|\geqslant\delta} |K_\epsilon(t)|dt] + \frac{L_1L_2C_1\|f_\epsilon^{(n+1)}\|_p}{(n+1)!}h^{n+1} \end{aligned} \tag{18}$$

Where h is the mesh size for x_n, L_1, L_2 is the Lipchiz constant, C_1 is a constant, n is the degree of interpolation degree, p and p' are conjugate components.

Proof. For the first term,

$$\begin{aligned} \|\hat{f} - f_\epsilon\|_p &\leqslant L_1 \cdot \|\hat{f}\circ g_1(x) - f_\epsilon \circ g_1(x)\|_p \\ &\leqslant \frac{L_1C_1\|f_\epsilon^{(n+1)}\|_p}{(n+1)!} \max\{g_1(x_i) - g_1(x_{i-1}), i = 1, 2...N\}^{n+1} \\ &\leqslant \frac{L_1L_2C_1\|f_\epsilon^{(n+1)}\|_p}{(n+1)!}h^{n+1} \end{aligned}$$

Notice that in the second inequality, we just apply the basic error estimate for piecewise interpolation. This can be found in many books about *Numerical Analysis*[1].

This finishes the first term.

For the second term[2],

$$\hat{f}(x) = \hat{f}(x)\int_{R^n} K_\epsilon(t)dt = \int_{R^n} \hat{f}(x)K_\epsilon(t)dt$$

So

$$\begin{aligned} |\hat{f}(x) - f_\epsilon| &= |\int_{R^n} [\hat{f}(x-t) - \hat{f}(x)]K_\epsilon(t)dt \\ &\leqslant \int_{R^n} |\hat{f}(x-t) - \hat{f}(x)| \cdot |K_\epsilon(t)|^{\frac{1}{p}} \cdot |K_\epsilon(t)|^{\frac{1}{p'}}dt \end{aligned} \tag{19}$$

In (2), we apply Holder's Theorem for the term $\int_{R^n} K_\epsilon(t)dt$ with conjugate components p and p'. Now we raise (2) to p^{th} power and integrate with respect to x, we get

$$\begin{aligned} \int_{R^n} |\hat{f}(x) - f_\epsilon(x)|^p dt &\leqslant \|K\|_1^{\frac{p}{p'}} [\int_{R^n}\int_{R^n} |\hat{f}(x-t) - \hat{f}(x)|^p |K_\epsilon(t)| dt] dx \\ &= \|K\|_1^{\frac{p}{p'}} \cdot \int_{R^n} |K_\epsilon(t) \cdot [\int_{R^n} |\hat{f}(x-t) - \hat{f}(x)|^p |dx] dt \end{aligned} \tag{20}$$

So $\|f_\epsilon - \hat{f}(x)\|_p^p \leqslant \|K\|_1^{\frac{p}{p'}} \cdot \int_{R^n} |K_\epsilon(t) \cdot \|\hat{f}(x-t) - \hat{f}(x)\|_p^p dt$

Notice that in (3), we can change the order of integration because the function $(f(x-t) - f(x))^p (K_\epsilon(t))$ is in $L^p(R^n \times R^n)$. And this comes from the fact that both $f(x)$ and $K_\epsilon(t)$ are in $L^p(R^n \times R^n)$.

For $\delta > 0$, write

$$\begin{aligned} \int_{R^n} |K_\epsilon(t)| \cdot \|\hat{f}(x-t) - \hat{f}(x)\|_p^p dt &= \int_{|t|<\delta} + \int_{|t|\geqslant\delta} \\ &= A_{\epsilon,\delta} + B_{\epsilon,\delta} \end{aligned}$$

Continuity of L^p space says: If $f \in L^p,\ 1 \leqslant p < \infty$, then

$$\lim_{|h|\to 0} \|f(x+h) - f(x)\|_p = 0$$

So $\forall\eta > 0,\ \exists\delta > 0,\ s.t.\ if\ |t| < \delta,\ \|f(x+h) - f(x)\|_p^p < \eta.$
Then

$$A_{\epsilon,\delta} \leqslant \eta \int_{|t|<\delta} |K_\epsilon(t)| dt \leqslant \eta \cdot \|K\|_1 \ for\ all\ \epsilon \tag{21}$$

Moreover, $\|\hat{f}(x-t) - \hat{f}(x)\|_p^p$ is a bounded function. By Minkowski's Inequality,

$$\begin{aligned} \|\hat{f}(x-t) - \hat{f}(x)\|_p^p &\leqslant (\|\hat{f}(x-t)\|_p + \|\hat{f}(x)\|_p)^p \\ &= (2\|\hat{f}(x)\|_p)^p \end{aligned} \tag{22}$$

So, $B_{\epsilon,\delta} \leqslant (2\|\hat{f}(x)\|_p)^p \cdot \int_{|t|\geqslant\delta} |K_\epsilon(t)| dt \to 0$ as $\epsilon \to 0$ for some fixed $\delta > 0$.
From (4) and (5),

$$\|f_\epsilon - \hat{f}\|_p \leqslant \|K\|_1^{\frac{p}{p'}} \cdot [\eta\|K\|_1 + (2\|f\|_p)^p \int_{|t|\geqslant\delta} |K_\epsilon(t)| dt]$$

This finishes the second term. □

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