Coding, Channel Capacity, and Noise Resistance in Communicating with Chaos

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Recent work has considered the possibility of utilizing symbolic representations of controlled chaotic orbits for digital communication. We argue that dynamically a coding scheme usually leads to trajectories that live on a nonattracting but noise-resisting chaotic saddle embedded in the chaotic attractor. We present analyses and numerical computation which indicate that the channel capacity of the chaotic saddle has a devil-staircase-like behavior as a function of the noise-resisting strength. The implication is that nonlinear digital communication using chaos can yield a substantial channel capacity even in a noisy environment. [S0031-9007(97)04462-1]

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Recently, it has been demonstrated that chaotic systems can be manipulated, via arbitrarily small time-dependent perturbations, to generate controlled chaotic orbits whose symbolic representation corresponds to the encoding of a desirable message [1]. Specifically, imagine a chaotic power oscillator that generates a large amplitude signal consisting of an apparently random sequence of positive and negative peaks. By associating a positive peak with a **1**, and a negative peak with a **0**, one obtains a signal that yields a binary sequence. It was shown how the use of small controls could cause the signal to follow an orbit whose binary sequence encodes an arbitrary message [1,2]. An advantage of this type of communication strategy is that the nonlinear chaotic oscillator that generates the wave form for transmission can remain simple and efficient, while all the necessary electronics controlling encoding of the signal remain at the low-powered microelectronic level. Moreover, since the chaotic dynamics can be recovered from a chaotic signal, which in principle can be noisy, by using standard dynamical data analysis techniques, communicating with chaos is also more robust and better behaved against channel noise [2].

A critical issue in communicating with chaos is how to select a proper coding scheme by which any message can be encoded in the chaotic signal. Imagine the two-symbol (**0** and **1**) case and consider n-bit symbol sequences. For a nonlinear oscillator that generates a chaotic attractor, if the dynamics corresponds to a Bernoulli shift, there are then 2^n possible symbol sequences. The number of allowed symbol sequences, however, in most naturally occurring chaotic oscillators is less than 2^n: the rules allowing for specific ones form the grammar of the particular dynamics, meaning that there are forbidden symbol sequences. Since the grammar of a natural chaotic oscillator is quite complicated, it is difficult to design a code which takes a full advantage of the dynamics by excluding only the forbidden symbol sequences. Thus, in practice, one chooses a code that restricts the grammar so that only a subset of all the allowed symbol sequences is utilized. Since all the allowed symbol sequences correspond to the original chaotic attractor in the phase space, the symbol sequences only allowed by the code corresponds to a chaotic saddle embedded in the attractor.

To illustrate how to design a code and to understand its consequence for the corresponding dynamics, we consider the Lorenz system: $\dot{x} = 10(y - x)$, $\dot{y} = x(28 - z) - y$, $\dot{z} = xy - (8/3)z$, which can be physically realized by an electric circuit [3]. Let z_n be a maximum of the state variable $z(t)$. Then the successive maxima can be described by a one-dimensional, single maximum, nondifferentiable map $z_{n+1} = f(z_n)$. The chaotic attractor in the three-dimensional phase space $\{x(t), y(t), z(t)\}$ corresponds to a one-dimensional chaotic attractor in the phase space of the discrete map $f(z)$. The natural partition for defining a good symbolic dynamics is the critical point z_c where $f(z_c)$ is maximum. A trajectory point with $z < z_c$ ($z > z_c$) bears symbol **0** (**1**). Now suppose we choose a code in which four **0**'s in a row are forbidden in any n-bit sequence, where $n > 4$. This restriction may be imposed by the dynamics. In the symbol space, the code removes all n-bit sequences that have four or more **0**'s in any location. In the z phase space of the map $f(z)$, the restriction imposed by the code corresponds to forbidden gaps. In fact, the forbidden infinite-bit sequences with no more than three **0**'s in a row correspond to gaps in the z space which are forbidden by the code. These gaps are open and dense. This means that the symbol sequences allowed by the code correspond to a dynamics restricted to a Cantor set in the z space, as shown in Fig. 1. This Cantor set corresponds to an invariant chaotic saddle embedded in the original chaotic attractor in the $\{x, y, z\}$ space. Note that, in general, it is in fact advantageous to utilize the chaotic saddle such as the one shown in Fig. 1 for communication because the symbolic encoding (and decoding) is immune to small noise. Say the system is in a noisy environment. If no code restriction is used to encode messages, a bit error (i.e., **0** becomes **1** or vice versa) could occur when the trajectory comes close to the partition point z_c since noise

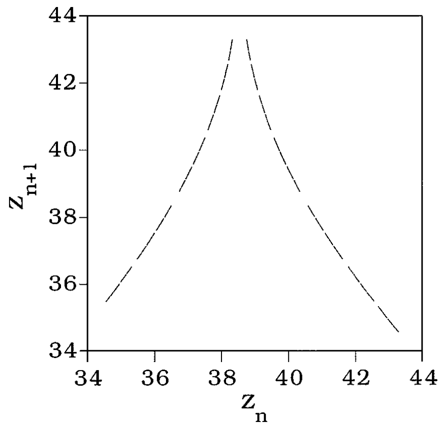

FIG. 1. A 10 000 point trajectory of the Lorenz map on a noise-resisting chaotic saddle embedded in the chaotic attractor, corresponding to imposing the grammatical restriction, "no four **0**'s in a row."

could kick the trajectory across z_c. The possibility for bit error due to noise can be substantially reduced when a code such as the one yielding Fig. 1 is chosen since there is a noise gap about the partition point z_c.

To give some illustrative examples of an encoding message in the restricted chaotic signals, say we wish to communicate the message "BEAT ARMY!" in the following ASCII format by using the Lorenz attractor:

$$\overbrace{1000010}^{B}\overbrace{1100101}^{E}\overbrace{1100001}^{A}\overbrace{1110100}^{T}\overbrace{0100000}^{\text{space}}$$
$$\overbrace{1000001}^{A}\overbrace{1110010}^{R}\overbrace{1101101}^{M}\overbrace{1111001}^{Y}\overbrace{0100001}^{!}\,.$$

To transmit the message subject to the "no four zeros in a row" code, a simple way is for the transmitter to insert a buffer bit "$\underline{1}$" after three zeros in a row, regardless of the message bit that follows. Thus, the encoded message becomes

$$\overbrace{1000\underline{1}010}^{B}\overbrace{1100101}^{E}\overbrace{11000\underline{1}01}^{A}\overbrace{1110100}^{T}\overbrace{01000\underline{1}00}^{\text{space}}$$
$$\overbrace{1000\underline{1}001}^{A}\overbrace{1110010}^{R}\overbrace{1101101}^{M}\overbrace{1111001}^{Y}\overbrace{01000\underline{1}01}^{!}\,.$$

Furthermore, if the original message contains the block 0001, with three zeros in a row, the modified block is $000\underline{1}1$. Thus, the receiver can recover the original message simply by stripping a one after every block of three zeros. Since for the Lorenz attractor, its intrinsic grammar is already included in the rule no four zeros in a row, the message BEAT ARMY! can now be transmitted using the Lorenz circuit [3] by utilizing small control methods outlined in Ref. [1]. One may also consider a more severe restriction such as "no three zeros in a row," which corresponds to a larger gap across the partition line. In this case, the encoded binary sequence looks like

$$\overbrace{100\underline{1}00\underline{1}10}^{B}\overbrace{1100\underline{1}101}^{E}\overbrace{1100\underline{1}00\underline{1}1}^{A}\overbrace{1110100\underline{1}}^{T}\overbrace{0100\underline{1}00\underline{1}0}^{\text{space}}$$
$$\overbrace{100\underline{1}00\underline{1}01}^{A}\overbrace{11100\underline{1}10}^{R}\overbrace{1101101}^{M}\overbrace{111100\underline{1}1}^{Y}\overbrace{0100\underline{1}00\underline{1}1}^{!}\,.$$

Since more buffer bits are needed, the transmission rate will be slower, but the code is more immune to noise as the noise-resisting gap is wider.

By overrestricting the code, one does not take full advantage of the natural chaotic dynamics produced by the power oscillator, resulting in a communication system with reduced channel capacity. Dynamically, the channel capacity is quantified by the topological entropy of the invariant set in which the message-carrying trajectory lies [4]. Thus, in order to optimize the channel capacity, one must design a code that maximizes the topological entropy. Since a chaotic saddle is an invariant subset embedded in the original chaotic attractor, the topological entropy of the saddle is generally smaller than that of the attractor. While a larger size of the noise gap about the partition renders the symbolic dynamics more robust against noise, the resulting chaotic saddle possesses smaller topological entropy. This is due to the fact that widening the noise gap corresponds to increasing the grammatical restrictions on the permissible codes in the symbol dynamics representation. Thus, the purpose of this Letter is to argue that chaotic saddles embedded in a chaotic attractor can be noise resisting but also rich information sources for communication. Furthermore, we show that there is a trade-off between noise resistance and channel capacity when designing a code for communication applications. In particular, an appropriate code restriction exists which generates a noise resisting chaotic saddle that optimizes the channel capacity versus the noise resistance. We also present strong evidence indicating that the topological entropy of the chaotic saddle is a nonincreasing, devil's-staircase-like function of the noise-gap size.

To facilitate a systematic numerical computation and analysis of the topological entropy of chaotic saddles, we make use of the logistic map $f(x) = rx(1 - x)$ which captures the essential dynamics of the single-maximum maps arising in physical situations such as the Lorenz system. Consider the case where the map exhibits a chaotic attractor. We assign a symbol **0** (**1**) to the trajectory of $x < 1/2$ ($x > 1/2$) (the critical point $x_c = 1/2$ is the partition). For a chaotic saddle with a primary gap of size s centered at x_c, its topological entropy h_T is less than that of the chaotic attractor. As s increases, the number of allowed symbol sequences cannot increase. Figure 2(a) shows h_T versus s for $r = 4$. To generate this figure, we compute, for each s, a long trajectory of 10^7 points on the chaotic saddle by using the PIM-triple (proper interior maximum) algorithm [5] and then count $N(n)$, the number of possible symbol sequences of length n. The topological entropy is given by $h_T = \lim_{n\to\infty} \frac{\ln N(n)}{n}$. In practice, we plot $\ln N(n)$ versus n for n up to, say, 20. The slope of such a plot is approximately the topological entropy h_T. From Fig. 2(a), we see that h_T is apparently a nonincreasing function of s. A striking phenomenon is that there seem to be numerous plateau regions of s in which h_T remains approximately constant. We find numerically that these plateau regions appear to exist on all scales in s. The

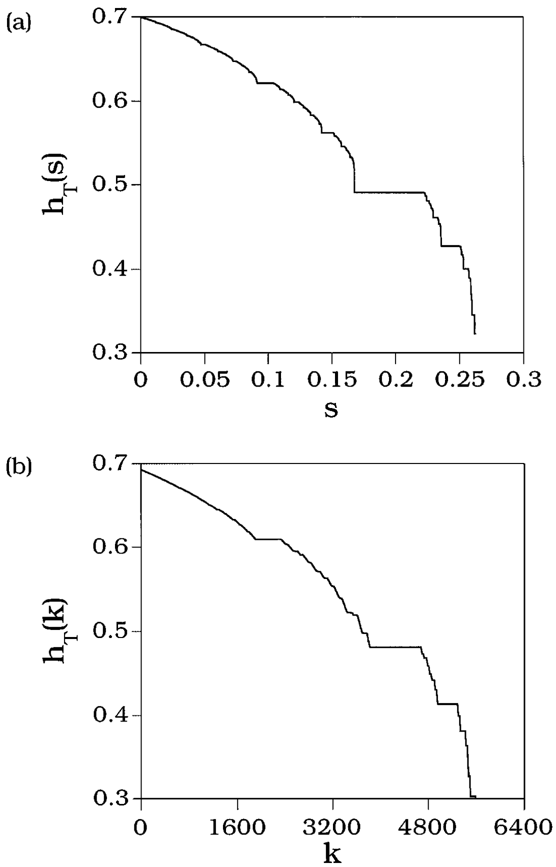

FIG. 2. (a) Numerical computation of the topological entropy h_T versus the size of the noise-resisting gap s for the logistic map at $r = 4$. (b) Theoretical prediction using 14-bit symbol sequences. Some details of the theoretical calculation differ from the numerical result in (a), due to the fact that only 14-bit sequences are used. The theory nonetheless predicts correctly the devil's staircase structure.

set of s values at which h_T changes seems to have arbitrarily small Lesbegue measure in the parameter s. Similar behavior is also observed for other parameter values of the logistic map with well developed chaos. These results thus strongly suggest that the function of h_T versus s is a devil's staircase.

A feature of the h_T-versus-s function, which is common to chaotic parameter values of r [exemplified by Fig. 2(a)] and of practical importance, is that h_T decreases only slightly in a wide region when the noise-gap size increases from zero initially. In Fig. 2(a), for example, the topological entropy of the chaotic attractor is $\ln 2 \approx 0.69$. As s is increased from 0 to 0.1, h_T decreases from $\ln 2$ to about 0.62, a rather small decrease. But $s = 0.1$ means that the symbolic dynamics on the chaotic saddle is robust against noise of amplitude about 5×10^{-2}. Thus, with only incremental loss in the channel capacity, the symbolic dynamics on the chaotic saddle is immune to external noise of relatively large amplitude [6].

We now give the theoretical justification for the devil staircases seen in Fig. 2(a). Our analysis is based on successive approximations of the grammar of the symbolic dynamics corresponding to the chaotic map, using a sequence of *transition matrices*. For a one-dimensional single-maximum map of the form $x_{n+1} = f(x_n)$, a good symbolic dynamics can be defined by dividing the phase space into two disjoint but dynamically connected subsets S_0 and S_1 by using a Markov partition [7]. The orbit $\{x_i\}_{i=0}^{\infty}$ of an initial condition x_0 defines an itinerary sequence $\{\sigma_i\}_{i=0}^{\infty}$ through the partition, $\sigma_i = g(x_i) = \mathbf{0}$ if $x_i \in S_0$ and $\sigma_i = g(x_i) = \mathbf{1}$ if $x_i \in S_1$. Let Σ be the set of *all* possible infinite symbol sequences of the symbols **0** and **1**. An initial condition x_0 has an itinerary sequence written as $\sigma = \sigma_0.\sigma_1\sigma_2\sigma_3\ldots \in \Sigma$. The Bernoulli-shift map B: $\Sigma \rightarrow \Sigma$, defined by $B(\sigma) = B(\sigma_0.\sigma_1\sigma_2\sigma_3\ldots) = \sigma_1.\sigma_2\sigma_3\sigma_4\ldots$, evolves symbol sequences in Σ. Thus, the dynamics on the chaotic attractor can be represented by the dynamics of $B|_{\Sigma'}$ (the Bernoulli shift map restricted to a subshift Σ'), where $\Sigma' \subset \Sigma$, is a closed and invariant subset of Σ. A finite n-bit symbol sequence $\sigma_0.\sigma_1\ldots\sigma_{n-1}$ identifies all points in Σ which agree in their first n bits. The n-bit symbol sequence corresponds to a neighborhood, or bin, in the phase space by a change of variable. As n is increased, these bins become increasingly refined.

The grammar of the subshift Σ' can be defined by the collection of all permissible (or alternatively, forbidden) transitions between n-bit words, under the action of the Bernoulli-shift map restricted to Σ'. The n-bit bins are generated by the sequence of preimages of the critical point x_c: $\{x_c, f^{-1}(x_c), f^{-2}(x_c), \ldots, f^{-(n-1)}(x_c)\}$. Note that when the map is not everywhere two onto one, some $f^{-i}(x_c)$ will not exist, and consequently, there is an illegal i-bit word. A subshift Σ' of finite type has a grammar which is representable by a finite list of forbidden n-bit words. In this case, the grammar is represented by a 2^n node directed graph, or equivalently, by a $2^n \times 2^n$ transition matrix, A_n [7]. The Bernoulli-shift map permits at most two arrows into and two arrows out of each n-bit node, corresponding to the choice of shifting in a **0** or a **1** bit from any state. For the case of the full-shift grammar $B|_{\Sigma}$ in which there are no forbidden n-bit words, each row and each column of A_n has at most two nonzero entries. The topological entropy of a subshift of finite type can be computed directly as the natural logarithm of spectral radius of the generating transition matrix [8], $h_T = \ln[\rho(A_n)]$. Hence, the topological entropy of a subshift of infinite type (corresponding to $n \rightarrow \infty$) can then be computed in terms of the limit of spectral radii of a sequence of transition matrices $\{A_n\}$, which generate a sequence of approximating subshifts $\{\Sigma_{A_n}\}$ to Σ',

$$h_T(\Sigma') = \lim_{i \to \infty} \ln[\rho(A_i)]. \qquad (1)$$

This is the basis of our direct calculation of $h_T(M(s))$, for which *we know* the form of the continuously varying subshifts of $M(s)$, where $M(s)$ denotes the chaotic saddle with gap size s, embedded in the chaotic attractor $M(0) \equiv M$.

Given a single-maximum map, such as the logistic map, the key feature which allows us to apply Eq. (1) to calculate the topological entropy as a function of gap width s is the fact that we know the ordering of the n-bit itinerary bins ("Gray code" ordering) [9]. When formulating an n-bit word approximation of the subshift Σ', we construct the $2^n \times 2^n$ transition matrices ordered according to increasing Gray codes. For each n-bit approximation of the fullshift grammar B_Σ, each of the 2^n nodes has two entering arrows and two exiting arrows. Therefore the transition matrix A_n has exactly two ones in each row and column, and it follows that $h_T = \ln(\rho(A_n)) = \ln(2)$. A restriction on the grammar of Σ_{A_n} corresponds to a forbidden n-bit word. If the jth n-bit word is forbidden, then all transitions into and out of the jth node are also forbidden to preserve the invariance of the resulting subshift with respect to the Bernoulli shift. Hence, the corresponding transition matrix A_n' has all zero entries in the jth row *and* the jth column.

Widening the gap s is discretely approximated in terms of restricting the grammar of the corresponding n-bit subshift. Although, in general, we do not know *a priori* the locations and widths of each n-bit itinerary bin, in the case of single-maximum maps, we do know the order in which they are eliminated. Thus, we can compute the spectral radius of the transition matrices starting with the full shift grammar Σ_{A_n}. We eliminate pairs of n-bit words (by zeroing corresponding rows and columns), starting from the middle. At each step, we compute h_T of the evolving subshift directly from the spectral radius of the current transition matrix. Let k be the number of bins eliminated which scales monotonically to the gap width s. We expect topological entropy to be a monotonically nonincreasing function of k. Increased restrictions on the grammar leads to decreased channel capacity. Increasing n, the word size considered, better approximates the effect of continuously increasing the gap size s from $s = 0$; a small increase in s requires n large to account for a whole (small) bin which is eliminated. Figure 2(b) reveals a devil's-staircase-like function $h_T(k)$ for $n = 14$. As n is increased, more constant intervals, or "flat spots," are revealed; the invariance of the subshift has required us to eliminate all transitions away from an eliminated node, and often this may effectively eliminate other nearby nodes. In such a case, further widening of the gap, and hence elimination of the next node in the Gray ordering, causes no change because that node may have already been dynamically eliminated in a previous step. The devil's staircase structure arises from the fact that this mechanism occurs on all scales, to 2^n node directed graph representations of the symbol dynamics of the chaotic saddles $M(s)$, for all n, and the larger the n-bit word size considered, the smaller the flat spot will be.

The fact that the topological entropy of the chaotic saddle decreases only slightly in a range of gap sizes $(0, \Delta s)$ has important practical implications. Say the noise amplitude is $\Delta s/10$. Then the chaotic saddles with gap sizes in $(\Delta s/10, \Delta s)$ are immune to noise, yet their channel capacity is only slightly less than that of the original chaotic attractor. There are an infinite number of codes that can generate chaotic saddles with gap sizes in $(\Delta s/10, \Delta s)$. From the standpoint of channel capacity and noise resistance, these codes are therefore optimal. Similar results appear to hold for high-dimensional chaotic systems.

In conclusion, we have presented an analysis and numerical results which imply that practical coding schemes in communicating with chaos yield the utilization of chaotic saddles embedded in a chaotic attractor. This has several advantages: (i) Chaotic saddles can be noise resisting, (ii) the loss in the channel capacity is only incremental, and (iii) the specification of symbolic partition may be significantly simplified (particularly for high-dimensional systems). These advantages may make practically feasible communicating with chaos.

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