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COMMUNICATION AND SYNCHRONIZATION IN DISCONNECTED NETWORKS WITH DYNAMIC TOPOLOGY: MOVING NEIGHBORHOOD NETWORKS

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Abstract. We consider systems that are well modelled as networks that evolve in time, which we call *Moving Neighborhood Networks*. These models are relevant in studying cooperative behavior of swarms and other phenomena where emergent interactions arise from ad hoc networks. In a natural way, the time-averaged degree distribution gives rise to a scale-free network. Simulations show that although the network may have many noncommunicating components, the recent weighted time-averaged communication is sufficient to yield robust synchronization of chaotic oscillators. In particular, we contend that such time-varying networks are important to model in the situation where each agent carries a pathogen (such as a disease) in which the pathogen's life-cycle has a natural time-scale which competes with the time-scale of movement of the agents, and thus with the networks communication channels.

1. **Introduction.** Network dynamics has become a very important area in nonlinear studies because so many systems of interest have a natural description as a network. Examples include the internet, power grids, neural networks (both biological and other), social interactions, and many more. However, the preponderance of the work in complex networks does not allow for dynamic network topology [1, 2]. In the literature, one generally finds that either a static network is 'born', as in the small-world (SW) [6] and Erdos-Renyi [4] models, or that a network evolves into an otherwise static configuration, as is assumed in the Barabasi-Albert model of scale-free (SF) networks evolution [7]. In epidemic modeling and percolation theory, one considers the problem of certain links being knocked-out, but essentially as a static problem, since there is no possibility for links to reform within the theory. Fluid Neural Network (FNN) models provide one approach to incorporating local transient interaction effects into a variety of dynamical systems [15]. In the recently presented Coupled Map Gas (CMG) model [5], neighborhood coupling of motile elements, with coupling and state of the elements affecting future evolution of the system, provides a study of how such schemes support pattern formation

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among the elements. In the work of Stojanovski, Kocarev, et al. [8], on-off time-varying coupling between two identical oscillators is considered as a synchronization problem, with the very different from ours assumption that at each period of the connection, one of the variables is reset in an initial condition changing alteration. While the results of [8] must therefore be considered not completely related to ours, it is interesting to point out that in [8], a theorem is proved in which there exists a fast enough period T such that synchronization is asymptotically stable in the time-varying coupled case if it is asymptotically stable in the constantly coupled case. We would also like to point out follow-up work, [9] in which a spatio-temporal system, which is a ring lattice - a simple type of graph - is similarly controlled by sporadic coupling with initial condition resetting.

In this paper we consider a simple model that allows network links to follow their own dynamical evolution rules, which we consider a natural feature of many organic and technological networks, where autonomous agents meander or diffuse, and communication between them is an issue of both geography and persistence. For our concept problem, we focus on social interactions, such as *a disease propagating across a network of social contacts.* See Fig. 1. A suitable model should consider the disease life-cycle, which may be just a matter of a few weeks; an epidemic ensues only if agents connect within that time window. Thus, with such expiring messages, what matters is whom we have *recently* contacted. Said simply, we are not likely to catch the common cold *directly* from an old friend, who is sick, but whom we have not seen in many years and we will not see for more years still, since the disease runs its entire course in a much shorter time-scale than that of our contact period. We contend that any model which does not account for time-varying connections in a natural way cannot properly account for this ubiquitous concept.

When the time scale for network changes is of nearly the same order as the time scales of the underlying system dynamics, we believe that network evolution should be part of the model. Moreover, evolution of the network at these time scales becomes a key element in understanding the effective connectivity of the network with respect to these expiring messages. We will study such networks using a model which we introduce here and which we call *moving neighborhood networks* (MN). We then modify the basic MN structure by considering that some social connections survive even when the local neighborhood has changed, which we call the *moving neighborhood with friends* model (MNF).

We evolve (diffuse) positions of agents independently according to a dynamical system or stochastic process, linking those nodes that are within the same neighborhood. We assume that the system has an ergodic invariant measure, then we prove that the relative positions of the nodes, and hence their connectivity, is essentially random, but with a well defined *time average degree distribution.* Our model departs from CGM in that motion of the agents is independent of the dynamics on the network. Our focus is the communication characteristic of the evolving network.

A time-varying network presents a time-varying graph Laplacian. Since the phenomenon of synchronization of oscillators in a network relies on open communication channels in the network, and since it has been previously shown in the case of static networks that the spectrum of eigenvalues of the static graph Laplacian plays an essential role in determining synchronization of the graph-coupled oscillators [14] we develop an analysis using synchronization as a natural probe of the time-scale over which a message can traverse the network which is now time-varying. In particular, if we take the "oscillator" carried by each agent to be their personal disease life-cycle (say an SIR model for each agent), then it is easy to see that synchronization as a probe of communication is relevant for these dynamics with channel competing time-scales. With this in mind, however, we have chosen chaotic oscillator synchronization as a harder test of our formalism. In our development, we generalize the concept of a master stability function [14], and we define a "moving-average" graph Laplacian. Using the property of synchronization as a probe of connectivity, we show a most striking feature of such networks is that while at any fixed time, the network may be fractured into noncommunicating subcomponents, the evolving network allows communication of those messages which do not expire on time-scales at which a message can find paths between subcomponents; this point is made clear by the fact that MN and MNF networks admit surprisingly robust synchronization well below the threshold when the network has a giant component.

We consider two distinct dynamical systems: 1) the dynamical system which governs the network topology by diffusing agents corresponding to the network nodes, which we call the *network dynamics,* and 2) the network of the oscillators which run at each vertex, with coupling between them moderated by the instantaneous network configuration, which we call the *system dynamics.* See Fig. 1. The formalism of *master stability function* [14] (which assumes a fixed network) must be modified to consider evolving networks. Synchronization requires sufficient information flow, so complete paths must appear on time scales relevant to the *system dynamics.* We introduce the concept of a *moving average Laplacian* to quantify the connectivity associated with a time sensitive message that propagates on a partially connected but changing network.

2. **The Moving Neighborhood Network:** Our modeling goal is to capture some features of evolving social networks. See Fig. 1. Using an analogy of collaborators: often we work as part of a group at our place of employment, usually dealing with a small group of people (our neighborhood). People move from one job to another, so our neighborhood will frequently change, but only a little. However, if *we* move, our neighborhood changes significantly; we end up with a completely different group of coworkers. The time scale of the small changes is of the same order as might be required to tackle significant problems and therefore are relevant to the overall productivity of the group.

To capture this dynamic, we associate network nodes with an ensemble of points evolving under a flow, forming a time dependent network by linking nodes that are 'close.' Notationally, let,

$$\bar{\xi} = \{\xi_1, \ldots, \xi_n\}, \tag{1}$$

be a collection of n points in some metric space $\mathbf{M}$. We construct graph Ξ from $\bar{\xi}$ by associating a vertex with each element of $\bar{\xi}$. Vertices i and j of Ξ are assigned to be adjacent (connected by an edge) if,

$$|\xi_i - \xi_j| < r \implies i \leftrightarrow j, \tag{2}$$

where r is a parameter that defines the size of a neighborhood. Let,

$$\phi_t : \mathbf{M} \mapsto \mathbf{M}, \tag{3}$$

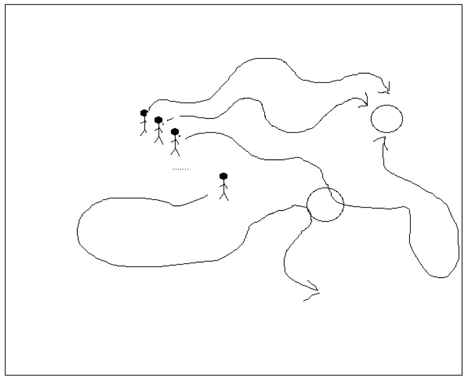

FIGURE 1. A charicature depicting N agents which wander ergodically as an ensemble according to a process Eq. (5). A time-varying proximity graph results according to Eq. (2), which we call a (MN) moving neighborhood network. The circles shown represent ϵ-close approach balls mentioned in Eqs. (2), (29) and (33). With the ergodic assumption, MN leads to Eq. (31) that the degree distribution is binomial, or asymptotically Poission for many agents. If furthermore, there is connection "latency," there results a scale-free structure as evidenced by Fig. 5. This is what we argue is a reasonable model to consider propagation of diseases and other quantities which have their own life-cycle time-scale which competes with the time-scale with which the agents move.

be the flow of some dynamical system on $\mathbf{M}$. From an initial ensemble,

$$\bar{\xi}^0 = \{\xi_1^0, \ldots, \xi_n^0\}, \tag{4}$$

we define an ensemble trajectory by

$$\bar{\xi}(t) = \Phi_t(\bar{\xi}^0) = \{\phi_t(\xi_1^0), \ldots, \phi_t(\xi_n^0)\}, \tag{5}$$

which in turn generates a graph $\Xi(t)$ that precribes a *network trajectory.* The flow ϕ_t may be governed by any discrete or continuous time diffusive process, either deterministic or stochastic. For brevity of presentation, we describe the MN process by using a deterministic, ergodic map (γ), representative of strobing a continuous diffusive system. Under suitable choice of γ, most ensembles will distribute according to some natural invariant density, ρ_γ, giving a well defined time-average network character.

3. **Simulation: a specific MN network.** Consider the following construction: Let $\mathbf{M} = \mathrm{T}^1$, the circle, and let

$$\gamma(x) := 1.43x - .43\frac{\lfloor 4x \rfloor}{4} \bmod 1. \tag{6}$$

This map, chosen primarily for illustrative reasons, has the following characteristics: (1) it is choatic, (2) transitive on the invariant set $[0, 1]$, (3) uniformly expanding, (5) with non-uniform invariant density, and (5) is discontinuous (so that a node may be moved to a distant neighborhood on one iteration). From a random initial condition for $\bar{\xi}$, we iterate past the transient phase so that the ensemble resembles the invariant density. We then construct the associated network for each iteration of map γ. Fig 2 shows the network constructed from five successive iterations, using $n = 28$ and $r = 0.09$. Note that from one iteration to the next, the connections associated with node 1 change very little. The reindexing and redrawing in the second row makes clear that the network is a neighborhood graph, though not all neighborhoods contain the same number of nodes. Note that for all but time $\tau + 2$, the graphs have a disconnected component.

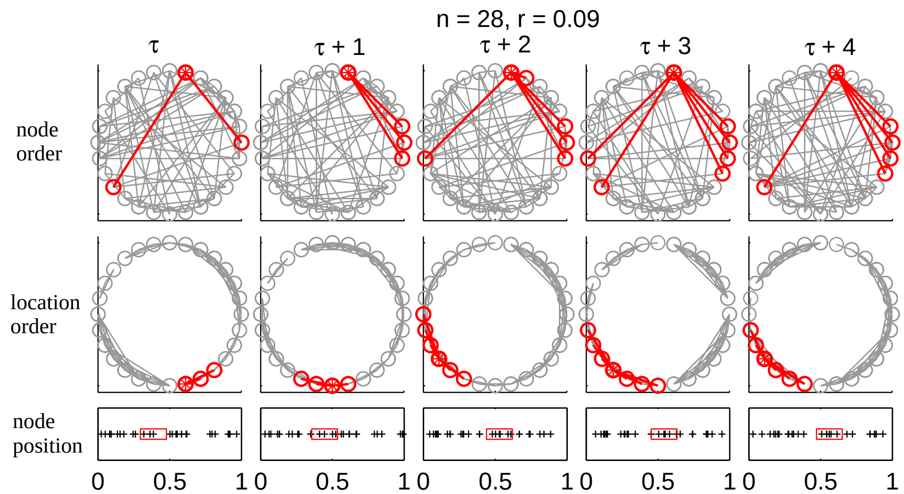

Figure 2. (Color online) An MN simulation using $n = 28$, $r = .09$, and map of (6). Five time steps are shown. The 1st row shows the network in node index order. The 2nd row is the same network, with the nodes positioned by ξ_i ordered to illustrate that they are neighborhood graphs. The (bold) red portion of the network shows connections to node 1. The bottom row shows the ensemble distribution of $\bar{\xi}$. The relatively small $n = 28$ was chosen for artistic reasons (to more easily display the connections).

4. **Synchronization of Coupled Oscillators:** To explore the implications of an MN structure, we use synchronicity as a connectivity persistence probe of a network of n identical chaotic oscillators. We form a time dependant network, described by graph $G(t)$, consisting of n vertices $\{v_i\}$, together with the set of ordered pairs of vertices $\{(v_i, v_j)\}$ which defines the edges. The $n \times n$ adjacency matrix defines the edges, $A_{i,j}(t) = 1$ if there is an edge (v_i, v_j) at time t, and $= 0$ otherwise. The system of n oscillators is linearly coupled by the network as follows: Let the vector $\mathbf{x}_i \in \mathbf{S} = \mathbb{R}^p$ be the state vector for the ith oscillator and express the coupled system as

$$\dot{\mathbf{x}}_i(t) = f(\mathbf{x}_i(t)) + \sigma \sum_{j=1}^{n} L_{ij}(t) K \mathbf{x}_j(t), \tag{7}$$

where σ a control parameter, $L_{ij}(t)$ the element of the graph Laplacian,

$$L(t) = \mathrm{diag}\,(d) - A(t), \tag{8}$$

and K specifies which state vector components are actually coupled. If we assume the network is MN, we have a dynamical system flowing on $\mathbf{M}^n \times \mathbf{S}^n$. Specifically, we consider the Rössler attractor with $a = 0.165, b = 0.2, c = 10.0$, which exhibits a chaotic attractor with one positive Lyapunov exponent [11]. Coupling the n systems through the x_i variables, the resultant system is given by,

$$\begin{array}{l} \dot{x_i} = -y_i - z_i - \sigma \sum_{j=1}^{n} L_{ij}(t) x_j \\ \dot{y_i} = x_i + a y_i \\ \dot{z_i} = b + z_i(x_i - c). \end{array} \tag{9}$$

Then the question of whether the oscillators will synchronized is reduced to whether one can find a value for σ such that the synchronization manifold is stable.

5. **Known results for static networks.** For a fixed network, necessary conditions for synchronization are well described by the approach in [12, 14], summarized as follows: The graph Laplacian matrix L has n eigenvalues, which we order as,

$$0 = \theta_0 \leq \ldots \leq \theta_{n-1} = \theta_{max}. \tag{10}$$

Using linear perturbation analysis, the stability question reduces to a constraint upon the eigenvalues of Laplacian:

$$\sigma \theta_i \in (\alpha_1, \alpha_2) \quad \forall i = 1, \ldots, n-1, \tag{11}$$

where α_1, α_2 are given by the *master stability function* (MSF), a property of the oscillator equations. For σ small, synchronization is unstable if $\sigma\theta_1 < \alpha_1$; as σ is increased, instability arises when,

$$\sigma \theta_{max} > \alpha_2. \tag{12}$$

By algebraic manipulation of (11), one can show that if,

$$\frac{\theta_{max}}{\theta_1} < \frac{\alpha_2}{\alpha_1} =: \beta, \tag{13}$$

then there is some coupling parameter, σ_s, that will stabilize the synchronized state. For some networks, no value of σ satisfies (11). In particular, since the multiplicity of the zero eigenvalue defines the number of completely reducible subcomponents, if $\theta_1 = 0$, the network is not connected, and synchronization is not stable. However, even when $\theta_1 > 0$, if the spread of eigenvalues is too great, then synchronization may still not be achievable.

6. **Numerical explorations of MN behavior:** Consider a system of $n = 100$ agents wandering on the chaotic attractor of the Duffing equation,

$$q'' = q - q^3 - .02q' + 3 \sin t,$$

whose driven frequency is commensurate with the natural frequency of the Rössler system, $\omega \approx 1$. We construct an MN network based on that system by assuming network coupling between node i and node j if their separation in phase space ($\mathbb{R}^2$) is less than r. A Rössler system is associated with each node, and the oscillators are x-coupled in accordance with the evolving network. When we set $r = 1.1$, we find that the ratio $\frac{\lambda_{max}}{\lambda_1}$ is almost always greater than β, and there are even short time periods when the network is not connected. With the Rössler systems starting from a random initial condition, Fig 3 shows a plot of $x_i(t)$ for the coupled system, which shows that *despite the weak instantaneous spatial connectivity of the network, the* *oscillators synchronize.* The bold curve illustrates the systems's approach to the synchronization state by graphing

$$\Delta(t) = \frac{1}{n}\sum_{i=1}^{n} |x_i(t) - \bar{x}(t)| + |y_i(t) - \bar{y}(t)| + |z_i(t) - \bar{z}(t)|,$$

where,

$$(\bar{x}(t), \bar{y}(t), \bar{z}(t)) = \frac{1}{n}\sum_{i=1}^{n}(x_i(t), y_i(t), z_i(t)), \tag{14}$$

estimates the synchronization manifold. The exponential decay of $\Delta(t)$ seems to indicate asymptotic stability of the synchronized state. Our interpretation is that the rapidly changing laplacian allows for a temporal connectivity that augments the spatial to allow sufficient communication between nodes to support synchronization. Results are similar for other ergodic systems used to control agent flow, such as $\gamma(x)$ in Eq. (6).

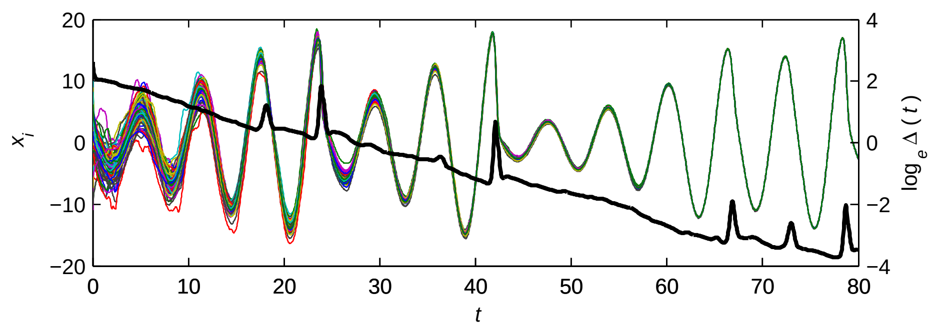

FIGURE 3. (Color online) MN Network with $n = 100$ nodes, $r = 1.1$, and coupling constant $\sigma = .2$. The agents wander according to the chaotic Duffing equation, $q'' = q - q^3 - .02q' + 3\sin t$. The x-coordinate of each oscillator is plotted vs. time. The bold line is Δ, providing an estimated deviation from the synchronization manifold.

7. **Analysis and conjectures:** The simulations show that although the synchronized state may be linearly unstable at each instant, the MN network can still synchronize. The instantaneous interpretation is that an ensemble of conditions near the manifold is expanding in at least one direction, but is generally contracting in many other directions. When the network reconfigures, the expanding and contracting directions change, so points in the ensemble that were being pushed away at one instant may be contracted a short time later. If there is sufficient volume contraction and change in orientation of the stable and unstable subspaces, the MN network can achieve asymptotic stability. In the following paragraph, we give some mathematical basis of the above by considering a simple linear system which is analagous to the variational equation of the synchronization manifold.

Consider the n dimensional initial value problem

$$\dot{z} = A(t)z, \qquad z(0) = z_0, \tag{15}$$

where,

$$A(t) = \sum_i \chi_{[iT,(i+1)T]}(t) A_i, \tag{16}$$

is a piecewise constant matrix, i an integer, and T constant. For narrative simplicity here, assume A_i is a diagonal matrix,

$$A_i = \mathrm{diag}\{\lambda_{i1}, \ldots, \lambda_{in}\}. \tag{17}$$

Since diagonal matrices commute, we may write the time $t_k = Tk$ solution to (15) as

$$z(t_k) = e^{\int_0^{t_k} A(\tau)d\tau} z_0 = e^{(A_0+\cdots+A_{k-1})T} z_0. \tag{18}$$

The fundamental solution matrix is diagonal with entries,

$$\lambda_j = e^{s_{jk}}, \tag{19}$$

with,

$$s_{jk} = \sum_{i=0}^{k-1} \lambda_{ij} T, \tag{20}$$

and each j can be associated with a coordinate direction in, R^n. Stability of the origin is ensured if s_{jk} is bounded above for all j and k. If in addition, $s_{jk} \to -\infty$, then the origin is asymptotically stable. Suppose the A_i's are chosen ergodically from a distribution such that for all i,

$$tr(A_i) = \sum_{j=0}^{k-1} \lambda_{ij} < \epsilon < 0. \tag{21}$$

Moreover, assume that the positive and negative eigenvalues are distributed ergodically along the diagonal elements of A_i. Then the time average (over i) must be the same as the spacial average (over j) of the eigenvalues, which implies that with probability 1, s_{jk} is bounded above and,

$$s_{jk} \to -\infty. \tag{22}$$

Since,

$$\det(\Phi(t_2, t_1)) = e^{\int_0^t tr(A(\tau))d\tau} < 1, \tag{23}$$

we have that the system is volume contracting.

8. **Assessing connectivity.**

Numerical simulations of the MN model indicate that synchronization can occur even when the network fails criteria of (11) *at every instant in time.* Apparently, the temporal mixing creates an average connectedness that allows the network to support synchronization. A logical conjecture is that connectivy could be assessed by examining the long-time average of the Laplacian of the network graph. If we assume ergodicity of the network dynamics, the long-time average of the laplacian is simply a scalar multiple of the Laplacian associated with a complete graph (all nodes connected), regardless of the size of the neighborhood and the mixing rate. It is known [13] that if the coupling is all to all, then synchronization can be stabilized. However, we can find instances with small neighborhoods and/or slow mixing such that there is no value of coupling constant which stabilizes the synchronization manifold. Therefore, we conclude that neither the instaneous nor the long time average Laplacian can accurately capture the connectivity of the MN network.

We conjecture that the inability for some networks to synchronize can be viewed as a lack of information carrying capacity within the network. A reasonable first guess is to assume that the information decays exponentially in time. We propose that an appropriate quantification of the average connectiviy is given by the **Moving Average Laplacian,** which we introduce here and define as the solution to the matrix initial value problem,

$$\dot{C}(t) = L(t) - \eta C(t), \quad C(0) = L(0), \tag{24}$$

where the coefficient η allows for variation of time scale within the system. Essentially, $C(t)$ is exponentially decaying to the current state of the network. We solve (24) to write

$$C(t) = e^{-\eta t}\left(C(0) + \int_0^t e^{\eta\tau} L(\tau) d\tau\right). \tag{25}$$

Since we are primarily interested in systems where the time scale of network evolution is commensurate with the time scale of the dynamics on the network, we generically assume $\eta = 1$.

Our desire with the Moving Average Laplacian, $C(t)$, is to describe the connectivity in a way that accounts for the temporal mixing. $C(t)$ has the property that if the mixing of the nodes is very slow compared to the system dynamics, its value will be nearly the same as the instantaneous connectivity, approximately equivalent to a sequence of fixed networks. However, if the mixing is very fast relative to system dynamics, then $C(t)$ will approximate the long time average, and the network connectivity is as if the network were complete. These asymptotic properties are consistent with intuition. We offer the moving average Laplacian, with its time-scale weight η, as the essential mathematical object in our study, and the use of synchronization as a probe of connectivity is meant to naturally illustrate this assertion, through the role of its spectrum.

Our definition of Moving Average Laplacian is indepedent of the particular system dynamics operating on the network, with the goal of describing the connectivity of moving networks without regard to specific application. To illustrate that there is some utility in this definition, we revisit our probe of connectivity — synchronization of chaotic oscillators. Since the instantaneous network has the property that,

$$\frac{\lambda_{max}}{\lambda_1} > \beta, \tag{26}$$

there is no value of σ that will allow us to satisfy the criteria of (11). At issue, then, is how does one choose a value for the coupling constant?

Consider the following naive approach: we estimate,

$$\lambda_1^* = E[\lambda_1(C(t))], \tag{27}$$

and,

$$\lambda_{max}^* = E[\lambda_{max}(C(t))], \tag{28}$$

and then use λ_1^* and λ_{max}^* with (11) to determine an appropriate choice for σ to achieve stable synchronization on a particular MN network. To examine the utility of this approach, we investigated four MN systems, two with the Duffing nodes moving at normal speed, and two with the nodes moving three times faster than normal. We define *synchronization exponent,* ν, to be the average slope on the graph of $\ln \Delta(t)$ for a small perturbation from the synchronized state. We examine ν as a function of coupling constant, where a negative value for ν represents an exponential approach to the synchronization manifold. We illustrate the results in Fig 4. For each curve, the bolded region shows those values of σ for which the Moving Average

Laplacian predicts a stable manifold. We note that the stability property in this range has been correctly predicted, but that the estimate is conservative, in the sense that the synchronization may remain stable for coupling values far outside that range.

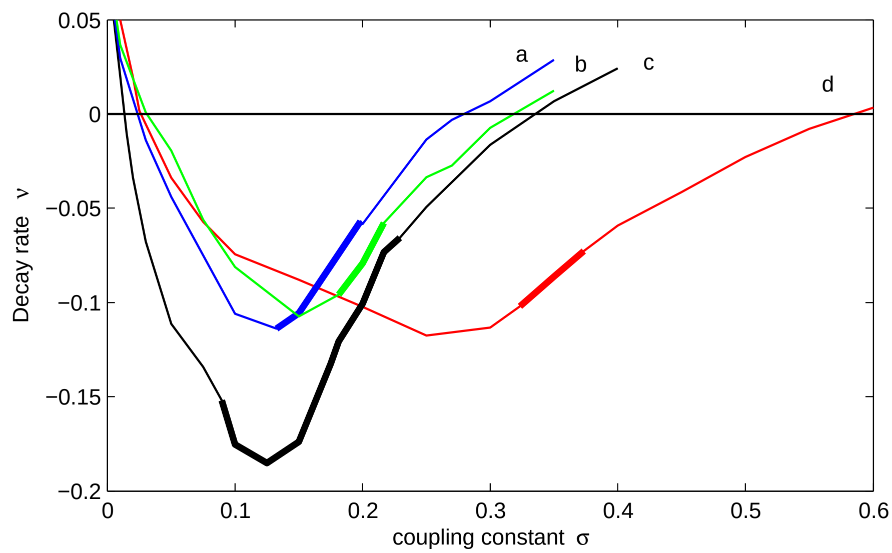

FIGURE 4. (Color online) Graphs of synchronization exponent ν as a function of σ. All systems used $n = 100$ nodes. Curve (a): $r = 1.2$, network an normal speed. Curve (b): $r = 1.1$, network at normal speed. Curve (c): $r = 1.1$, network at 3x speed. Curve (d): $r = .75$, network at 3x speed. The bold region on each curve indicates those values of σ for which the Moving Average Laplacian predicts a stable manifold.

We should not expect that the Moving Average Laplacian would provide precise criteria for synchronization, because our "naive" approach is fundamentally in error. The MSF approach to analysis of a network is derived based on a fixed network, whereas $C(t)$ still represents an evolving network. (We note that for curve (d) of Fig 4, the approach gave a very conservative estimate, which coincides with the fact that the behavior of that system is most dependent upon the mixing of the system, since the network with $r = .75$ generally has more than three disconnected components.) We recognize that there are techniques that should allow precise analysis of the synchronization behavior of MN networks, which will, of necessity, be significantly more complicated than the MSF. However, our goal with the Moving Average Laplacian was not to predict synchronization, but rather to quantify the connectivity. Because we were able to exploit this quantification to aid in choosing a stabilizing coupling parameter leads us to believe that the quantification may have utility in other areas of network analysis that rely on the spectrum of the laplacian, and that further investigation is warranted.

9. **Time-Average Scale-Free Network.** The main thrust of this modeling effort is to show that it is useful to consider evolving networks. The underlying time average degree distribution remains very flexible, including possibility of the scale-free distribution seen so frequently in many applications, [1, 2]. The basic MN network generates a binomial degree distribution, seen easily as follows. The probability

$$\begin{aligned} p(x,\epsilon) &\equiv \\ &= P(\text{agent-}j \text{ at position } y \text{ at least } \epsilon\text{-close to agent-}i, \text{ at } x : y \in B_\epsilon(x)) \\ &= \int_{B_\epsilon(x)} d\mu(y), \end{aligned} \tag{29}$$

(by assuming the network has the ergodic invariant measure $\mu(x)$). The 'long-run' probability that i and j coincide to within ϵ is

$$p(\epsilon) = \int p(x,\epsilon) d\mu(x), \tag{30}$$

where $p \equiv p(\epsilon)$ is a function of ϵ, as above. Therefore, the time-average degree distribution of MN is the binomial,

$$P_{p(\epsilon)}(k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k}, \tag{31}$$

which is asymptotically Poisson for $n >> 1$, or $p << 1$.

A time-averaged scale-free network requires a substantially heavier tail than the basic MN model. Thus motivated, and also considering that social connections, once formed, have certain persistence or memory, we model that some agents "stay in touch," continuing to communicate for some period after they are no longer neighbors. We formulate the following modification to MN, which we call Moving Network with Friends, or MNF: To each agent we associate a random "gregarious factor,"

$$g_i = U(0,1). \tag{32}$$

As with MN, a new link is made between agents i and j whenever,

$$|x_i - x_j| < \epsilon. \tag{33}$$

However, once formed, we introduce latency as follows: At each time step T after

$$|x_i - x_j| > \epsilon, \tag{34}$$

we break the link $i \leftrightarrow j$ iff a uniform random,

$$q = U(0,1), \tag{35}$$

variable satisfies,

$$q > F(g_j, g_i) = 1 - \sqrt{g_j g_j}, \tag{36}$$

where there is tremendous freedom in choosing F depending upon the application, but we have chosen a specific form as matter of example here. The exponential latency creates the power-law tail in the degree distribution, as shown in Fig. 5. The early rise left of the maximum follows since our model still forms connections according to the binomial distribution of MN, but now they are broken more slowly. For large k, we find empirically that,

$$P(k) \sim k^{-\alpha} \text{ with } \alpha \approx 2. \tag{37}$$

An MNF, since it provides additional connectivity, has more robust synchronization properties than an MN network with the same neighborhood size, r.

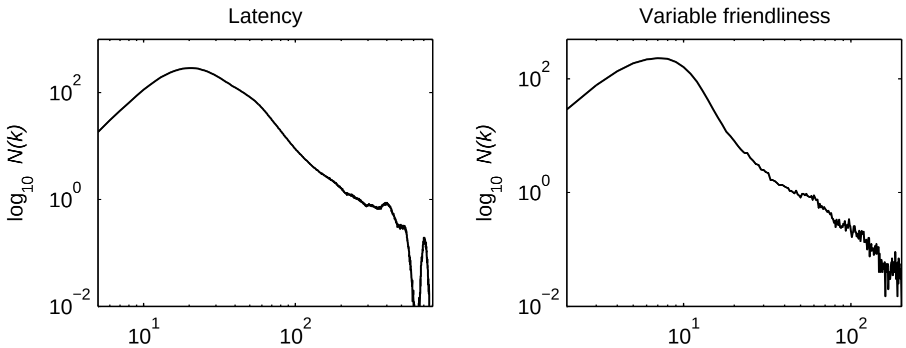

FIGURE 5. Either exponential latency (a) or exponential neighborhood size (b) can generate scale-free average distributions.

It easy to formulate other MN-type models which produce a scale free structure, and we mention one more which we find sufficiently applicable. One can model that some nodes are "friendlier" than others by defining the neighborhood of node i to be of size r_i, where r_i need not be the same for every node. An power-law distribution of r_i would also generate a time-average scale-free network.

10. **Conclusions and Direction:** In many real processes in which information propagation in ad hoc networks (such as disease spread, where the infective information may survive within an agent on the order of just weeks), the recent network connections play a crucial role in the dynamic behavior of the system. Thus we have been motivated to study time-evolving networks, which may more accurately describe the relevant dynamics. Our MN and MNF models provide a first attempt at developing such models, basing the network upon diffusing agents communicating within geographic neighborhoods and with established "friends." The numerical simulations in this paper show that global patterns (synchronization) are possible in these models, *even when the network is spatially disconnected.* We are developing a rigorous analysis of the moving average Laplacian to support our empirical work on how it captures the connectivity of evolving networks. Under the very general assumptions of ergodic network dynamics of the agents movements, we have proven the concept of an average degree distribution, and we have further shown that adding natural latency to network connectionism leads to the widely observed phenomenon of scale-free degree distribution, but now in a time-averaged sense, which is our new concept. We expect these models to widely provide insight into relevant issues regarding swarming, flocking and other physical and technological ad hoc cooperative and emergent behavior, particularly if one expects the flock to act in some fashion that achieves a goal separate from the coordinated movement. We believe the basic MN model can also be useful to understand the related control theoretic issue [16] of observability and controllability in the situation where agents are trying to coordinate some control action which is a fast moving process, but the communications channels are themselves time-varying; this is still open and important area of control systems in ad hoc networking.

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