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Mostly Conjugacy of Stochastically Perturbed Dynamical Systems

Rana D. Parshad

King Abdullah University of Science and Technology
Applied Mathematics and Computational Science
Thuwal 23955-6900
Kingdom of Saudi Arabia

Jiongxuan Zheng, Erik Bollt and Joseph D. Skufca

Clarkson University
Department of Mathematics & Computer Science
Potsdam, NY 13676, USA

Abstract

We study the conjugacy between two dynamical systems. We consider the case when one system is perturbed via an additive noise term. We lay down the rigorous notion of the commuter in such a setting and investigate its convergence properties. To aid our analysis we make extensive use of transfer operator methods. We also use the so called random operators from probabilistic functional analysis.

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1 Introduction

A central issue in the applied sciences is that of reduced order modeling. It is often difficult, if not impossible, to differentiate upon first inspection, a seemingly good model from perhaps a better one. The questions a modeler has to keep in mind are manyfold: What are the salient features of the physical process that should be captured by a model? What dynamical properties of the underlying process should be mimicked? What can perhaps be discarded? The answers in large part depend on the viewpoints, and thus the subsequent choices made by the modeler. However, there is also a well known formal means of comparison. In the dynamical systems literature two systems are considered to be the the same if there exists a conjugacy between them. This is essentially a homeomorphic change of coordinates that allows us to change freely between the two systems. However, for most real world applications this is seldom observed. No pragmatic model captures all the features of the process it represents. Thus a physical process and a "representative" model for it, are never quite conjugate. Once you accept that this discrepancy will almost always occur in most real world modelling applications, it becomes imperative to investigate the issues therein. This endevour seems to be important not only from a dynamical systems point of view, but from the more general standpoint of modeling in science.

The works of Bollt and Skufca, [BS07], [BS08], have investigated some of these issues in great detail. In these works they develop a systematic methodology to compare systems which are not quite cojugate, thus coining the phrase "mostly conjugacy". They lay down rigorously the notion of the commuter function, in such a setting. The commuter now departs from having the standard properties of a homeomorphism. Bollt and Skufca thus associate to this object a defect measure associated with how much it fails to be a homeomorphism. They prove the convergence of this object and derive a robust algorithm for its numerical computation. In [BS10] they give a symbolic dynamical interpretation to the commuter. The work of Zheng et al., [ZBS10], rigorously justifies these computations by proving various regularity results concerning the commuter functions considered therein. Zheng also makes various error estimates concerning the numerical approximation of the commuter function via the algorithm presented in [BS08].

A central theme of the above mentioned works is that they have all focused on a deterministic setting. In the current manuscript we will study the conjugacy between two dynamical systems, when one of these is perturbed via a stochastic term. Our hope is to lay down a rigorous first step in the direction of "mostly conjugacy" to stochastically perturbed dynamical systems. The foremost goal of this manuscript is to address the question, "How do you compare noisy systems, in the sense of conjugacy?". We hope to provide a reasonable answer to this question, detailing the nuances involved in the process, and discussing how we overcome them.

The manuscript is organised as follows. We first present a definition of the random commuter. Next we extend this analysis to cases where there is external noise. We deal with the cases of uniform noise and normal noise and present convergence results therein. Furthermore, we also present a numerical approximation of the random commuter and provide figures for this object.

This hopefully is and aid to sharpen geometric intuition. We also present an alternate view of the commuter function, as a **random** operator, and present further convergence results, in this light. Lastly we present some concluding remarks.

2 Preliminaries

Consider the following dynamical systems,

$$g_1(x) : X \to X, \tag{1}$$

$$g_2(y) : Y \to Y. \tag{2}$$

Recall that the dynamical systems are called conjugate if there exists a homeomorphism

$$f : X \to Y, \tag{3}$$

such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{g_1} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{g_2} & Y \end{array}$$

The above in particular implies that the following equation holds pointwise for all $x \in X$,

$$f(g_1(x)) = g_2(f(x)). \tag{4}$$

Our aim in the current manuscript is to adopt the above methodology to the case where one of the dynamical systems is forced by noise. In particular, we want to characterise the commuter in such a setting. It is intuitive that in the random setting we would wish to evolve densities, instead of individual points. This will lead naturally to our definition of a random commuter, which is the next order of business. We begin by constructing the requisite phase spaces. Assume now that instead of X our phase space is given by $L^1(X)$. Here $X = [0,1]$. Formally we will confine our attention to

$$D = \left\{ \rho \in L^1(X) : \int_X |\rho(x)| dx = 1, \rho \geq 0 \right\}. \tag{5}$$

D is thus a space consisting of densities on X. Furthermore instead of Y our phase space will be given by $L^1(Y)$ where $Y = [0,1]$. Again we are essentially considering

$$D^{'} = \left\{ \rho \in L^1(Y) : \int_Y |\rho(y)| dy = 1, \rho \geq 0 \right\}. \tag{6}$$

Instead of points $x \in X$ moving under the action of the dynamics of g_1, and points $y \in Y$ moving under the action of the dynamics of g_2, we have densities $\rho(x) \in D$, and $\rho(y) \in D^{'}$ that are evolved under the action of Frobenius-Perron operators. Note however, that $L^1(X)$ does not formally include the external noise. To proceed we must resolve this. Lets pause for a moment, and consider the following definition from [L91].

Definition 2.1 (Continuous Random Dynamical system) *A measurable random dynamical system on the measurable space (X, β) over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t \in T})$ with time T is a mapping*

$$\phi : \mathbb{T} \times \Omega \times X \to X, \ (t, \omega, x) \longmapsto \phi(t, \omega, x), \tag{7}$$

with the following properties

(i) Measurability: ϕ is $\beta(\mathbb{T}) \otimes \mathcal{F} \otimes \beta$- measurable.
(ii)Cocycle property: The mapping $\phi : X \to X$, form a cocycle over θ.

The point we want to emphasize is that under external stochastic forcing (or modeling error if you will), it becomes useful to visualise a random dynamical system as a movement on the bundle $\Omega \times X$. Here the source of the randomness is present in Ω, and takes values in $L^1(\Omega)$, and the dynamics take values in $L^1(X)$. In the light of the above we introduce the skew product $L^1(X \times \Omega) = L^1(X) \times L^1(\Omega)$ to represent our phase space. This enables us to incorporate the stochastic forcing into $L^1(X)$, by defining $L^1(X \times \Omega)$ as the right phase space. To reiterate, the noise is considered to be from external sources, be it pure randomness, modeling error or any combination therein. The transport of the densities are then handled formally by the stochastic Frobenius-Perron operators

$$P_{g_1}\rho(x) : L^1(X \times \Omega) \to L^1(X), \tag{8}$$

$$P_{g_2}\rho(y) : L^1(Y) \to L^1(Y). \tag{9}$$

With this in mind we can make the following definition.

Definition 2.2 *Consider the following dynamical systems*

$$g_1(x_n) = S_1(x_n) + \xi_n(\omega), \tag{10}$$

$$g_2(x_n) = S_2(x_n). \tag{11}$$

Where $\xi_n(\omega)$ is a stochastic forcing term with some prior distribution. Furthermore assume the Frobenius-Perron operators for the systems are given by

$$P_{g_1}\rho(x) : L^1(X \times \Omega) \to L^1(X), \tag{12}$$

$$P_{g_2}\rho(y) : L^1(Y) \to L^1(Y). \tag{13}$$

The random commuter $f : L^1(X) \to L^1(Y)$, between the two dynamical systems g_1 and g_2, is a density pointwise, in the sense that the following holds

$$f(P_{g_1}\rho(x)) = P_{g_2}f(\rho(x)). \tag{14}$$

This can be viewed as the following diagram commuting

$$\begin{array}{ccc} L^1(X \times \Omega) & \overset{P_{g_1}}{\to} & L^1(X) \\ \downarrow_f & & \downarrow\downarrow_f \\ L^1(Y) & \overset{P_{g_2}}{\to} & L^1(Y) \end{array}$$

Remark 2.3 *The two arrows on the right are a representation for many to oneness. Each for a particular realisation of the noise. So in essence, for any particular realisation of the noise ω, the Frobenius-Perron operator P_{g_1} acts on a density $\rho(x,.) \in L^1(X \times \Omega)$, to produce a new density $P_{g_1}(\rho(x,.)) \in L^1(X)$. This density is carried by the commuter f to a new density $f(P_{g_1}(\rho(x,.))) \in L^1(Y)$. Along the other leg the commuter f acts on the density $\rho(x,\omega)$ and carries it to $L^1(Y)$. Here it is picked up by the Frobenius-Perron operator P_{g_2}, and carried to $L^1(Y)$. The action of the commuter f can be viewed as a change of coordinates, following the standard theory of change of variables for densities. Here the jacobian of the transformation is essentially the Radon-Nikodym derivative, as highlighted in the next subsection.*

2.1 The Case Of Uniformly Distributed Noise

Consider the following two dynamical systems g_1 and g_2 as defined below.

$$\begin{aligned} & g_1(x,\xi_x) \\ = \; & 2(1-2\epsilon)x + (\xi_x+1)\epsilon, \; 0 \le x < 1/2, \\ = \; & 2(1-2\epsilon)(1-x) + (\xi_x+1)\epsilon, \; 1/2 \le x \le 1. \end{aligned} \tag{15}$$

$$\begin{aligned} & g_2(y) \\ = \; & 2y, \; 0 \le y < 1/2, \\ = \; & 2(1-y), 1/2 \le y \le 1. \end{aligned} \tag{16}$$

We assume that ξ_x is uniformly distributed in the interval unit interval $[0,1]$. We wish to show the convergence of the Frobenius-Perron operator under g_1. Note the operator under consideration now contains a stochastic kernel. This can be written out explicitly following [LM91]. Let (X, A, μ) be a measure space. For any $f \in D$ we have

$$P_{g_1} f(x) = \int_X K(x,y) f(y) dy. \tag{17}$$

Here $K(x,y)$ is a stochastic kernel that satisfies

$$K(x,y) \ge 0, \tag{18}$$

and

$$\int_X K(x,y) dx = 1. \tag{19}$$

For our purposes

$$K(x,y) = g(x - S(y)) \text{ where } g(x) = 1_{[0,1]}(x) \text{ is the density of } \xi. \tag{20}$$

We begin by introducing some definitions and then recalling a theorem from [LM91].

Definition 2.4 *Let (X, A, μ) be a measure space and $P : L^1(X) \to L^1(X)$ a Markov operator. Then $\{P^n\}$ is said to be* ***asymptotically stable*** *if there exists a unique $f_* \in D$ such that $Pf_* = f_*$ and*

$$\lim_{n\to\infty} ||P^n f - f_*|| = 0 \text{ for every } f \in D. \tag{21}$$

Theorem 2.5 (Lasota & Mackey, 1991) *Let (X, A, μ) be a finite measure space and $P : L^1(X) \to L^1(X)$ a Markov operator. Assume there is a $p > 1$ and $K > 0$ such that for every density $f \in D$ we have $P^n \in L^p$ for sufficiently large n, and*

$$\limsup_{n\to\infty} ||P^n f||_p \leq K. \tag{22}$$

Then P is constrictive.

Definition 2.6 *A constrictive operator is asymptotically stable.*

Note that P being constrictive implies the convergence of $\{P^n\}$. We next state our convergence result via the following Lemma,

Lemma 2.7 *consider the following dynamical system perturbed stochastically*

$$\begin{aligned} & g_1(x, \xi_x) \\ = \ & 2(1-2\epsilon)x + (\xi_x + 1)\epsilon, \ 0 \leq x < 1/2, \\ = \ & 2(1-2\epsilon)(1-x) + (\xi_x + 1)\epsilon, \ 1/2 \leq x \leq 1, \end{aligned} \tag{23}$$

where ξ_n is i.i.d uniformly distributed in [0,1]. The stochastic Frobenius-Perron operator for this dynamical system converges to the stationary density of the system. That is

$$\underset{n\to\infty}{lim} ||P^n f - f_*|| = 0 \ for\ every\ f \in D. \tag{24}$$

Proof 2.8 *The form of the Frobenius-Perron operator is easily constructed via techniques from [LM91]. It is given by the joint density of f_n and ξ. That is*

$$f_{n+1}(x) = P_{g_1} f_n(x) = \int_{\mathbb{R}} f_n(y) g(x - S(y)) dy. \tag{25}$$

Note

$$\begin{aligned} & S(x) \\ = \ & 2(1-2\epsilon)x + \epsilon, \ 0 \leq x < 1/2, \\ = \ & 2(1-2\epsilon)(1-x) + \epsilon, \ 1/2 \leq x \leq 1. \end{aligned} \tag{26}$$

Here $g(x-S(y))$ is the density of the random variable $\epsilon\xi$. Since ξ is uniformly distributed in [0,1], equation 25 reduces to

$$\begin{aligned} & P_{g_1} f_n(x) \\ = & \int_{\mathbb{R}} f_n(y)\epsilon \ 1_{[S^{-1}(x),S^{-1}(x-1)]}(x-S(y))dy \\ = & \ \epsilon \int_{S^{-1}(x)}^{S^{-1}(x-1)} f_n(y)dy. \end{aligned} \tag{27}$$

Thus it follows that

$$\begin{aligned} & \limsup_{n\to\infty} |P_{g_1} f_n(x)|_2^2 \\ = & \limsup_{n\to\infty} |\int_{\mathbb{R}} f_n(y)\epsilon \ 1_{[S^{-1}(x),S^{-1}(x-1)]}(x-S(y))dy|_2^2 \\ = & \ |\epsilon \int_{S^{-1}(x)}^{S^{-1}(x-1)} \limsup_{n\to\infty} f_n(y)dy|_2^2. \end{aligned} \tag{28}$$

Note that here and else where in the text we adopt the convention that the L^2 norm is represented as follows $\int_\Omega |f(x)|^2 dx = |f|_2^2$. The limit can be switched with the integral via Lebesgue dominated convergence theorem. Note, that markov maps are dense in the space of piecewise linear maps, [BB01]. Thus it follows that there exists an N such that for $n \geq N$

$$|f_n(y) - f^*|_2 \leq \epsilon_1. \tag{29}$$

Where f^* is the density of a markov map, thus is given by a piecewise constant function

$$\limsup_{n\to\infty} \sum_{i=1}^{k} C_i 1_{[a_i,b_i]}(x). \tag{30}$$

The following can also be obtained via approximating f_n by a step function, since $f_n \in L^1(X)$ by definition, [ZW77]. In any event we obtain,

$$
\begin{aligned}
& |\epsilon \int_{S^{-1}(x)}^{S^{-1}(x-1)} \limsup_{n\to\infty} f_n(y)dy|_2^2 \\
= \; & |\epsilon \int_{S^{-1}(x)}^{S^{-1}(x-1)} (\limsup_{n\to\infty}(f_n(y) - \sum_{i=1}^{k} C_i 1_{[a_i,b_i]}(y)) + (\sum_{i=1}^{k} C_i 1_{[a_i,b_i]}(y))dy|_2^2 \\
\leq \; & \epsilon| \int_{S^{-1}(x)}^{S^{-1}(x-1)} \epsilon_1 dy|_2^2 + \epsilon| \int_{S^{-1}(x)}^{S^{-1}(x-1)} \sum_{i=1}^{k} C_i 1_{[a_i,b_i]}(x)|_2^2 \\
\leq \; & \epsilon\epsilon_1| \int_{S^{-1}(x)}^{S^{-1}(x-1)} dy|_2^2 + \epsilon K| \int_{S^{-1}(x)}^{S^{-1}(x-1)} dy|_2^2 \\
= \; & \epsilon(\epsilon_1 + K)|S^{-1}(x-1) - S^{-1}(x)|_0^1 \\
\leq \; & \epsilon(\epsilon_1 + K)(|S^{-1}(1)| + |S^{-1}(0)|) \\
\leq \; & \epsilon(\epsilon_1 + K)\max(1, \frac{1-\epsilon}{2(1-2\epsilon)}, -\frac{\epsilon}{2(1-2\epsilon)}) \\
\leq \; & 2K \max(\epsilon, \frac{\epsilon(1-\epsilon)}{2(1-2\epsilon)}, -\frac{\epsilon^2}{2(1-2\epsilon)}) \\
\leq \; & K, \; \textit{for } \epsilon \leq \frac{1}{4}. \qquad (31)
\end{aligned}
$$

Note that the K does not depend on ϵ or n, so the bound is uniform in ϵ and n. We can now take a lim sup *in the above, to conclude that*

$$\limsup_{n\to\infty} |P_{g_1} f_n(x)|_2^2 \leq K. \qquad (32)$$

Thus the Lemma is proved via application of Theorem 2.5, where the L^p space we have considered is L^2.

2.2 Numerical Approximation of the Random Commuter with Uniformly Distributed Noise

The goal of this short section is to outline our method of numerically approximating the random commuter, when a uniform noise is used. This complements the theoretical convergence results of the previous section and enables the reader to visualise the random commuter. One description of the random commuter, as discussed in section 2, is that of a random valued function. In this light, the commuter is viewed as a one parameter family of distributions, such that for each $x \in X$, the commuter $f_\xi(x)$ is a distribution. As an example, we consider a logistic map with additive (uniform) noise

$$g_1(x, \xi) = 3.75x(1-x) + \xi_x,$$

where the noise is independent of the particular value of x (and also, i.i.d under iteration of the map. Suppose we seek to model this noisy logistic map g_2, via a symmetric tent map of height 0.9. To numerically approximate the commuter that relates systems g_1 and g_2, we proceed as follows: Take a uniform grid U of x coordinates (on the interval $[0, 1]$, and take f_0 to be an identify map. Compute instances of the commuter values on that grid using the iterative scheme below, as in [BS08],

$$f_{k+1}(U) = \hat{g}_2^{-1} \circ f_k \circ g_1(U),$$

where the additive noise is i.i.d at each different point on the grid and at each iteration. We procede naively and simply assume that after discarding transient iterations (to allow for convergence of the random operator), for any fixed $x \in U$ the values $\{f_k(x)\}_k$ (for sufficiently large k) can viewed as coming from the distribution associated with the random commuter. Collecting these values over a large number of iterations allows us to treat that ensemble as approximating the distribution at each of the specified values of x. Figure **??** provides a visualization of the approximation of the complete set of marginal distributions via a "quadweb".

3 Random Fixed Point Analysis

We now change direction slightly and give another interpretation of the commuter function between two dynamical systems, one of which is randomly forced. A central theme of the current manuscript has been to understand the action of the commuter on densities. In this light, it is quite natural to view this transport of densities, via the commuter, as the action of a random operator, also known as an operator valued random variable. Just as in the deterministic setting, [BS08], we can then ask: does there exists a fixed point of this operator? An affirmative answer would reinforce our convergence results derived earlier. A fair bit is known about random operator theory. The study of random fixed points was first considered by the Prague school of probabilists in the 1950's. They have found much use in application since, including random differential equations in Banach spaces. Note, of particular interest might be to persons interested in modeling with differential equations, where the coefficients are random. There is a vast literature on probabilistic functional analysis, and the use of random operator theory. See [R72], [SL99]. [R72] in particular is a comprehensive text, that details the exhaustive scope of these probabilistic techniques. We will now recall some preliminaries as to best inform the coming analysis. We begin via the following definitions

Definition 3.1 *Let (X, A, μ) be a measure space. A mapping $T : X \times \omega \to$*

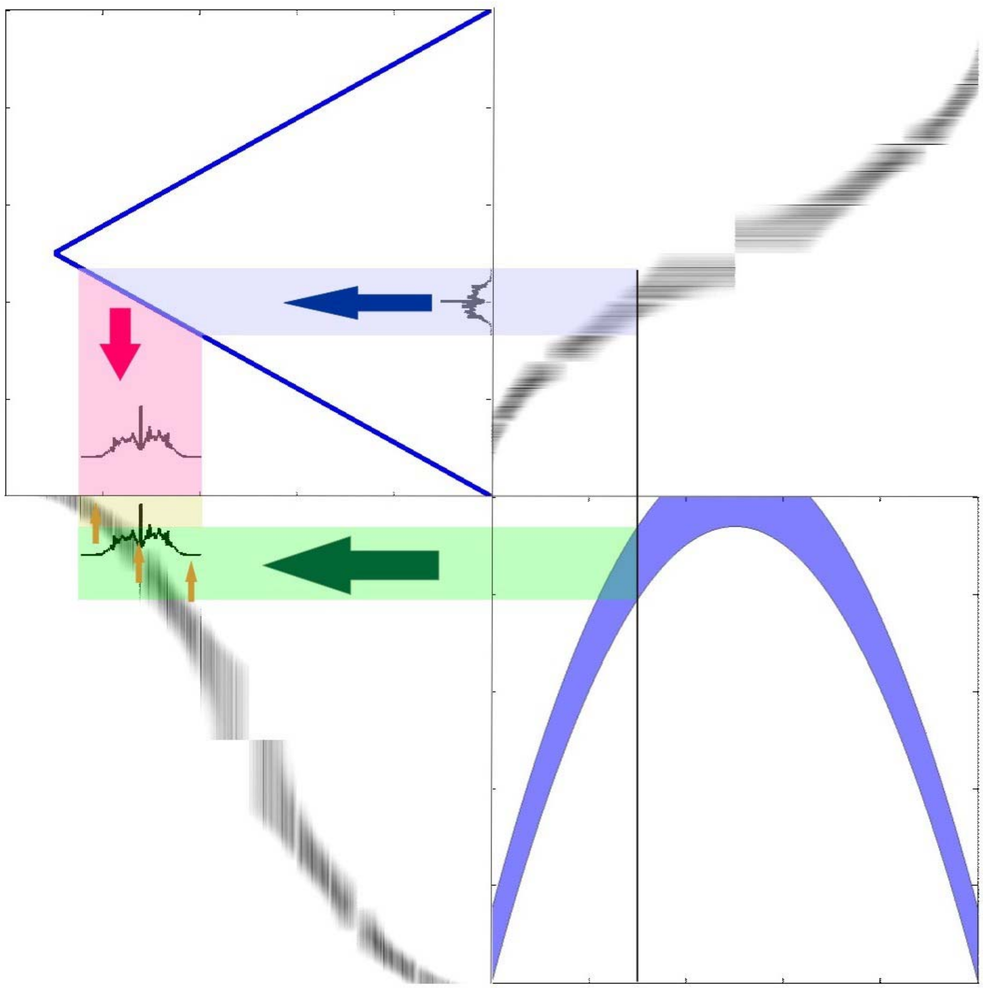

Figure 1: We perform some numerical simulations to visualise the random commuter best explained trough a "quadweb". This is our name for a graphical representation that allows us to visualise the action of the random commuter along both legs of the commutative diagram, in a single figure. See [BS08] for details. For this numerical experiment in the lower right panel we have a logistic map with $r = 3.75$ perturbed by a noise that is uniformly distributed in the unit interval. We see clearly that for a given value of x we can go upto the random commuter in the upper right panel, which yields a density that can then be evolved via the dynamics of the tent map with $r = 0.9$, ending up with a density seen in the pink column. We would end up with the same density if we evolved a density forward via the dynamics of the noisy logistic map, seen in the green column, and then acted on this via the random commuter.

X is called a random operator if for each fixed $x \in X$ the mapping the map $T(.,x) : \omega \to X$ is measurable.

Definition 3.2 *A measurable map $\xi : \Omega \to X$ is a random fixed point of a random operator $T : X \times \omega \to X$ if*

$$T(\omega, \xi(\omega)) = \xi(\omega), \text{ for each } \omega \in \Omega. \tag{33}$$

We recall the following theorem in the form best suited for our application.

Theorem 3.3 (Random Fixed Point Theorem) *Let T be a continuous random operator on $L^1([0,1] \times \Omega)$ to $L^1[0,1]$. Let λ_ξ be a real-valued random variable such that $\lambda_\xi < 1$ almost surely and*

$$||Tf_1 - \mathcal{T} f_2||_1 \leq \lambda_\xi ||f_1 - f_2||_1 \tag{34}$$

for every two functions $f_1,\ f_2 \in L^1[0,1]$. Then there exists an $L^1[0,1]$-value random function f_ξ, which is the unique fixed point of T, i.e.

$$Tf_\xi = f_\xi. \tag{35}$$

Our goal is to use the above to prove that the commuter function that we have considerd earlier, can be viewed as the fixed point of a random operator. This reinforces our result about the convergence of this object for various classes of noise, via stochastic Frobenius-Perron operator methods, that we considered in the previous section. Recall again the commutative diagram in the random setting

$$\begin{array}{ccc} L^1(X \times \Omega) & \overset{P_{g_1}}{\rightarrow} & L^1(X) \\ \downarrow_f & & \downarrow\downarrow_f \\ L^1(Y) & \overset{P_{g_2}}{\rightarrow} & L^1(Y) \end{array}$$

The above implies that the following holds

$$f(P_{g_1}(\rho(x))) = P_{g_2}(f(\rho(x))). \tag{36}$$

Clearly the commuter acts on densities in both legs of the diagram. Thus it is natural to view it as a random map, acting on densities, so as to bring in the afforementioned random fixed point theory. We first carefully define the right random operator in our current setting. We take our cue from earlier works, [BS08]. Recall in the deterministic setting f satisfies

$$f(g_1(x)) = g_2(f(x)). \tag{37}$$

Since we know what g_1 and g_2 are, a functional equation for f can be set up explicitly. We adopt the same methodology to the random setting. Consider the following figure illustrating a full tent map and a short map perturbed by noise

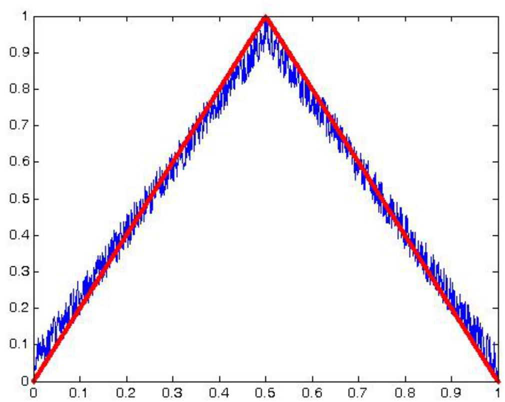

Figure 2: For this numerical experiment the systems we consider are a short tent map perturbed by a noise that is uniformly distributed in the unit interval, and a full tent map, that is with $r = 1$. In the figure seen above we consider a particular realisation of the noise. This is in contrast to the next figure, where an ensemble over all realisations is constructed.

We next describe these dynamical systems explicitly

$$\begin{aligned} & g_1(x, \xi_x) \\ = \;& 2(1-2\epsilon)x + (\xi_x + 1)\epsilon,\ 0 \le x < 1/2, \\ = \;& 2(1-2\epsilon)(1-x) + (\xi_x + 1)\epsilon,\ 1/2 \le x \le 1. \end{aligned} \tag{38}$$

$$\begin{aligned} & g_2(y) \\ = \;& 2y,\ 0 \le y < 1/2, \\ = \;& 2(1-y), 1/2 \le y \le 1. \end{aligned} \tag{39}$$

For any fixed density ξ, the following equation holds

$$f_\xi(g_1(x, \xi_x)) = g_2(f_\xi(x)). \tag{40}$$

Where $f_\xi : L^1(X \times \Omega) \to L^1(Y)$.
This yields

$$2f_\xi(x) = f_\xi(2(1-2\epsilon)x + (\xi_x+1)\epsilon,\ 0 \le x < 1/2, \tag{41}$$

$$2(1-f_\xi(x)) = f_\xi(2(1-2\epsilon)(1-x) + (\xi_x+1)\epsilon,\ 1/2 < x \le 1. \tag{42}$$

Thus we have a functional equation for $f_\xi(x)$,

$$f_\xi(x) = \frac{1}{2}f_\xi(2(1-2\epsilon)x + (\xi_x+1)\epsilon,\ 0 \le x < 1/2. \tag{43}$$

$$f_\xi(x) = 1 - \frac{1}{2}f_\xi(2(1-2\epsilon)(1-x) + (\xi_x+1)\epsilon,\ 1/2 < x \le 1. \tag{44}$$

We use the above to define a random operator as follows

$$\begin{aligned} & Tf_\xi(x) \\ = \ & \frac{1}{2}f_\xi(2(1-2\epsilon)x + (\xi_x+1)\epsilon,\ 0 \le x < 1/2, \\ = \ & 1 - \frac{1}{2}f_\xi(2(1-2\epsilon)(1-x) + (\xi_x+1)\epsilon,\ 1/2 < x \le 1. \end{aligned} \tag{45}$$

This follows $\forall f_1, f_2 \in L^1[0,1], 1 \le p < \infty$. We can now ask the following question: Does there exist a fixed point for the above operator? We were able to answer this in the affirmative in [BS08], via a contraction mapping argument. Here things are further complicated because of the presence of noise. Thus we have first introduced the relevant "noisy" operator theoretic setting, via the random fixed point Theorem. The existence of a random fixed point will imply convergence of the iterates of the noisy system, something we fully expect from our analysis via the Frobenius-Perron theory. Thus we state the following result

Lemma 3.4 *Consider the following dynamical system*

$$\begin{aligned} & g_1(x,\xi_x) \\ = \ & 2(1-2\epsilon)x + (\xi_x+1)\epsilon,\ 0 \le x < 1/2, \\ = \ & 2(1-2\epsilon)(1-x) + (\xi_x+1)\epsilon,\ 1/2 \le x \le 1. \end{aligned} \tag{46}$$

$$\begin{aligned} & g_2(y) \\ = \quad & 2y,\ 0 \leq y < 1/2, \\ = \quad & 2(1-y), 1/2 \leq y \leq 1. \end{aligned} \tag{47}$$

Where ξ_x is uniformly distributed in the interval $[0,1]$. The commuter function $f : L^1(X \times \Omega) \to L^1(Y)$ between the systems above, when viewed as an appropriate random operator, such as defined in 45, posesses a random fixed point.

Proof 3.5 *We consider the difference of Tf_1 and Tf_2 piecewise on $[0,1/2]$ and $[1/2,1]$ as defined.*

$$\begin{aligned} & ||Tf_1 - Tf_2||_{[0,1/2]} \\ = \quad & \int_{[0,1/2]} |Tf_1(x) - Tf_2(x)|dx \\ = \quad & \int_{[0,1/2]} |\frac{1}{2}f_1(2(1-2\epsilon)x + (\xi_x+1)\epsilon) - \frac{1}{2}f_2(2(1-2\epsilon)x + (\xi_x+1)\epsilon)|dx \\ = \quad & \int_{(\xi_x+1)\epsilon}^{(1-2\epsilon)+(\xi_x+1)\epsilon} |\frac{1}{2}f_1(y) - \frac{1}{2}f_2(y)|\frac{1}{2(1-2\epsilon)}dy \\ = \quad & \frac{1}{2}(\frac{1}{2(1-2\epsilon)})||f_1 - f_2||_{[(\xi_x+1)\epsilon,(1-2\epsilon)+(\xi_x+1)\epsilon]}, \end{aligned} \tag{48}$$

or

$$||Tf_1 - Tf_2||_{[0,1/2]} = \frac{1}{2}(\frac{1}{2(1-2\epsilon)})||f_1 - f_2||_{[(\xi_x+1)\epsilon,(1-2\epsilon)+(\xi_x+1)\epsilon]}. \tag{49}$$

This follows by setting $y = 2(1-2\epsilon)x + (\xi_x+1)\epsilon$, in the above. On the other hand, when $1/2 < x \leq 1$, we have

$$\begin{aligned} & ||Tf_1 - Tf_2||_{[1/2,1]} \\ = \quad & \int_{[1/2,1]} |Tf_1(x) - Tf_2(x)|dx \\ = \quad & \frac{1}{2}(\int_{(1-2\epsilon)+(\xi_x+1)\epsilon}^{(\xi_x+1)\epsilon} |\frac{1}{2}f_1(y) - \frac{1}{2}f_2(y)|\frac{1}{-2(1-2\epsilon)}dy) \\ = \quad & \frac{1}{2}(\int_{(\xi_x+1)\epsilon}^{(1-2\epsilon)+(\xi_x+1)\epsilon} |\frac{1}{2}f_1(y) - \frac{1}{2}f_2(y)|\frac{1}{2(1-2\epsilon)}dy) \\ = \quad & \frac{1}{2}(\frac{1}{2(1-2\epsilon)})||f_1 - f_2||_{[(\xi_x+1)\epsilon,(1-2\epsilon)+(\xi_x+1)\epsilon]}, \end{aligned} \tag{50}$$

or

$$||Tf_1 - Tf_2||_{[1/2,1]} = \frac{1}{2}(\frac{1}{2(1-2\epsilon)})||f_1 - f_2||_{[(\xi_x+1)\epsilon,(1-2\epsilon)+(\xi_x+1)\epsilon]}. \quad (51)$$

Again we set $y = 2(1-2\epsilon)(1-x) + (\xi_x + 1)\epsilon$ *in the above equation. We now add 49 to 51 to obtain*

$$\begin{aligned} & ||Tf_1 - Tf_2||_{[0,1]} \\ = & \frac{1}{2}(\frac{1}{1-2\epsilon})||f_1 - f_2||_{[(\xi_x+1)\epsilon,(1-2\epsilon)+(\xi_x+1)\epsilon]}. \\ \leq & \frac{1}{2}(\frac{1}{1-2\epsilon})||f_1 - f_2||_{[0,1]}. \end{aligned} \quad (52)$$

Thus we have

$$||Tf_1 - Tf_2||_{[0,1]} \leq \lambda ||f_1 - f_2||_{[0,1]}, \quad (53)$$

where

$$\lambda = \frac{1}{2}(\frac{1}{1-2\epsilon}). \quad (54)$$

We can choose $\epsilon < 1/4$*, so* $(\frac{1}{1-2\epsilon}) < 2$*. Thus we obtain*

$$\lambda = \frac{1}{2}(\frac{1}{1-2\epsilon}) < 1. \quad (55)$$

This completes the proof of the Lemma.

We next present a figure of this object as a visual aid to the prior analysis.

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4 Conclusion

In conclusion we have developed a new and rigorous definition of a random commuter. We have established its convergence via use of the Frobenius Perron theory, under pertubation via uniformly distributed noise and normally distributed noise. This result is seen in our numerical simulations also. Our

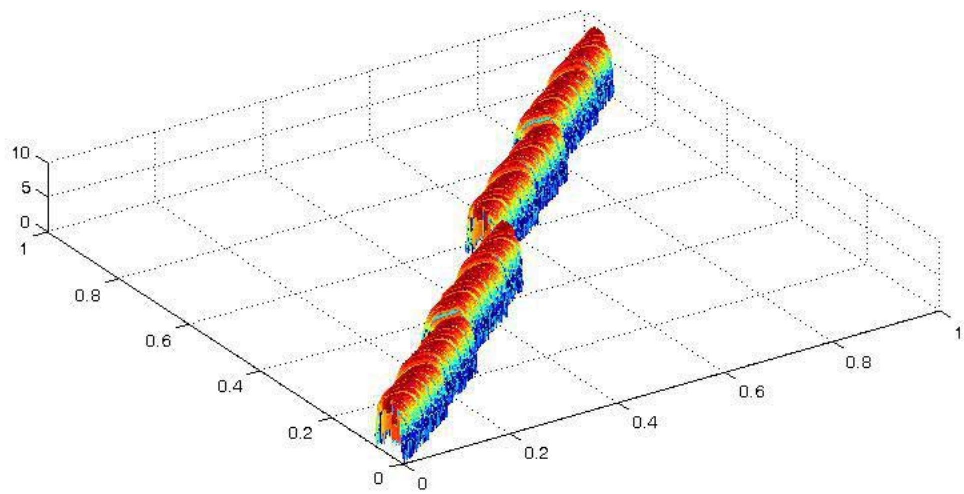

Figure 3: We perform some further numerical simulations to visualise the random commuter pictured above. Our aim here is to compare this object with our well understood notion of a deterministic commuter. For this numerical experiment the systems we consider are a short tent map perturbed by a noise that is uniformly distributed in the unit interval, and a full tent map. This yields the image seen in the figure above. Notice that for any x in the horizontal axis, we have a distribution across Y, in the vertical axis. Notice the pointwise blurring, but also the underlying fractal like function structure.

results concerning the commuter when viewed as a random operator reinforce these results. However various questions remain open at this point. Most of our analysis is confined to the case of one dimensional dynamical systems represented as maps. It would be interesting to consider how our methodologies apply when the underlying dynamical systems are flows. White noise perturbation in such a setting can also be considered. This would lead us to investigate the SDE case. The continuous version of the Frobenius-Perron operator might provide a valuable tool in this case. It is also interesting to try some of the above mentioned techniques when the commuters are not quite homeomorphic, which is getting at the true spirit of "mostly conjugacy". To this end we provide some details in Appendix B, where some of the associated difficulties can be circumvented via bringing in appropriately defined delta functions. When comparing flows, forced say via white noise, it might be possible to formulate certain large deviation principles. These might provide insight into how "conjugate" two flows are via comparison of their rate functions. Large deviations is one of the most active research areas in probability theory and might provide us with certain tools we could use in ways as alluded to earlier. It also seems worthwhile to calculate explicitly the random commuters for cases where the deterministic dynamics are far from conjugate (basically the case where one of the maps is to tall or to short) and compare these to the well understood deterministic cases. These and related questions are the subject of current investigation.

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