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Basis Markov Partitions and Transition Matrices for Stochastic Systems*

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Abstract. We analyze discrete-time dynamical systems subjected to an additive noise and their deterministic limit. In this work, we will introduce a notion by which a discrete-time stochastic system has something like a Markov partition for deterministic systems. For a chosen class of noise profiles, the Frobenius–Perron (FP) operator associated to the noisy system is exactly represented by a stochastic transition matrix of a finite size K. This feature allows us to introduce for these stochastic systems a basis Markov partition, defined herein, irrespectively of whether the deterministic system possesses a Markov partition or not. We show that in the deterministic limit, corresponding to $K \to \infty$, the sequence of invariant measures of the noisy systems tends, in the weak sense, to the invariant measure of the deterministic system. Thus, by introducing a small additive noise one may approximate transition matrices and invariant measures of deterministic dynamical systems.

Key words. stochastic dynamics, Markov partition, Frobenius–Perron operator, transition matrix

AMS subject classifications. 37H10, 37A05, 37M99, 37A30, 37A60, 15A99, 93E03

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1. Introduction.

Markov partitions for deterministic dynamical systems serve a central role for determining their symbolic dynamics [3, 4, 5] whose grammar is described by a finite sized transition matrix that generates a so-called sofic shift [6, 14]. The conditions for such a projection were defined by Bowen for Anosov hyperbolic systems [3, 4] and stated succinctly for interval maps as a partition whose elements are each a homeomorphism onto a finite union of its elements [3, 5]. We remark here that a defining property in both cases is that the set of characteristic functions defined over the elements of the Markov partition project the transfer operator exactly onto an operator of finite type; that is, a matrix results, whereas an infinite matrix would be expected for a non-Markov system. We argue here that this should be the defining property of any generalization of Markov partitions, that is, a set of basis functions which project the Frobenius–Perron (FP) operator exactly onto a finite-rank matrix with no

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residual. We present results here explicitly for random dynamical systems of the interval $[0, 1]$ and generalizations to the L-torus $[0, 1]^L$.

In physical literature on dynamical systems one often distinguishes a "natural" invariant measure of a hyperbolic system, which is stable with respect to an external noise [2, 7]. In mathematics this measure is known as the Sinai–Ruelle–Bowen (SRB) measure, and under certain assumptions one may rigorously prove its uniqueness [10]. Although the overall idea that adding the noise improves the convergence to the SRB measure is well known in the physics community, this work attempts to provide a more solid mathematical framework for this statement. In particular, for a certain class of the noise profiles, we are in position to characterize this convergence quantitatively.

First we recall the FP operator for a deterministic transformation. Associated with a discrete dynamical system acting on initial conditions, $\mathbf{x} \in M$ (say, a manifold $M \subset \Re^n$),

$$\begin{aligned} \tau : M &\to M, \\ x &\mapsto \tau(x), \end{aligned} \tag{1}$$

is another dynamical system over $L^1(M)$, the space of densities of ensembles of initial conditions

$$\begin{aligned} P_\tau : L^1(M) &\to L^1(M), \\ \rho(x) &\mapsto [P_\tau \rho](x). \end{aligned} \tag{2}$$

This FP operator (P_τ) is defined through a continuity equation [16],

$$\int_{\tau^{-1}(B)} \rho(x) dx = \int_B [P_\tau \rho](x) dx, \tag{3}$$

where B is a measurable subset of M, while PDF $\rho(x)$ belongs to $L^1(M)$. Equation (3) may be formally rewritten using the Dirac delta function:

$$[P_\tau \rho](x) = \int_M \delta(x - \tau(y)) \rho(y) dy. \tag{4}$$

This heuristic form is particularly suitable for further investigation of dynamical systems with additive noise; see (6).

Now consider the stochastically perturbed dynamical system

$$\begin{aligned} \tau_\nu : M &\to M, \\ x &\mapsto \tau(x) + \xi, \end{aligned} \tag{5}$$

where ξ is a random variable with PDF ν, which is applied once per each iteration. We assume that the realizations ξ_n of ξ added to subsequent iterations form an identical independently distributed (i.i.d.) sequence. The random part ξ is assumed to be independent of state x which we tacitly assume to be relatively small, so that the deterministic part τ has primary influence. The "stochastic FP operator" has a form similar to that of the deterministic case [16],

$$[P_{\tau_\nu} \rho](x) = \int_M \nu(x - \tau(y)) \rho(y) dy, \tag{6}$$

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where the deterministic kernel, the delta function in (2), now becomes a stochastic kernel describing the PDF of the noise perturbation. We will denote the stochastic FP operator as $P_{\mathcal{P}}$ below. In the case that the random map (5) arises from the usual continuous Langevin process, the infinitesimal generator of the FP operator for normal ν corresponds to a general solution of a Fokker–Planck equation [16]. The FP operator formalism is particularly convenient in that it allows for an arbitrary noise distribution ν to be incorporated in a direct and simple way. Within the formalism, we can also study multiplicative noise ($x \to \eta\tau(x)$, modeling parametric noise). The kernel-type integral transfer operator is $\mathcal{K}(x, y) = \nu(x/\tau(y))/\tau(y)$ for $x \in \Re^+$, which can then also be finitely approximated as described in the next section and usefully reordered to a canonical block reduced form. In more generality, the theory of random dynamical systems [1] clearly classifies those random systems which give rise to explicit transfer operators with corresponding infinitesimal generators, and there are well-defined connections between the theories of random dynamical systems and of stochastic differential equations.

The main aim of this work is to investigate a class of stochastically perturbed dynamical systems for which the FP operator is represented by a finite stochastic transition matrix of size N. Such dynamical systems will be called *basis Markov* in analogy to deterministic dynamical systems possessing a Markov partition [15, 25], for which the associated FP operator is finite. The deterministic limit of the stochastic system corresponds to the divergence of the matrix size. In this limit, $N \to \infty$, the sequence of invariant measures of the stochastic systems acting in the N-dimensional Hilbert space converges, in the weak sense, to the invariant measure of the corresponding deterministic system.

The paper is organized as follows. The Ulam–Galerkin method of approximating the infinite dimensional FP operator and the concept of the Markov partition for a deterministic system are reviewed in sections 2 and 3, respectively. In section 4 we introduce the notion of basis Markov stochastic systems, while in section 5 we analyze a particular example of random systems perturbed by an additive noise with cosine profile. In section 6 we construct a fairly general example of the transition densities satisfying our assumptions (20). The key result on convergence of the invariant measures for stochastic and deterministic systems is proved in section 7. A discussion of isospectral matrices used to describe the FP operator is relegated to the appendix.

2. Ulam–Galerkin's method: Approximating the infinite dimensional operator.

A Galerkin method may be used to approximate the FP operator by a Markov operator of finite rank. Formally, projection of the infinite dimensional linear space $L^1(M)$ results from discretely indexed basis functions $\{\phi_i(x)\}_{i=1}^{\infty} \subset L^1(M)$ onto a finite dimensional linear subspace generated by a subset of the basis functions,

$$\Delta_N = \text{Span}(\{\phi_i(x)\}_{i=1}^{N}). \tag{7}$$

This projection,

$$p : L^1(M) \to \Delta_N, \tag{8}$$

is realized optimally by the Galerkin method in terms of the inner product, which we choose to be integration:

$$(f, g) \equiv \int_M f(x)g(x)dx \quad \forall f, g \in L^2(M). \tag{9}$$

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Specifically, the infinite dimensional "matrix" is approximated by the $N \times N$ matrix,

$$A_{i,j} = ([P_\tau \phi_i], \phi_j) = \int_M [P_\tau \phi_i](x)\phi_j(x)dx, \quad 1 \le i, j \le N. \tag{10}$$

One approximates $\rho(x)$ through a finite linear combination of basis functions:

$$\rho(x) \simeq \sum_{i=1}^{N} d_i \phi_i(x). \tag{11}$$

The historically famous Ulam method [17, 27] for deterministic dynamical systems is equivalent to the interpretation for finding the fraction of the box B_i which maps by τ to B_j; the Ulam matrix is equivalent to the Galerkin matrix by using (10) and choosing the basis functions to be the family of characteristic functions

$$\phi_i(x) = \mathbf{1}_{B_i}(x) = \begin{cases} 1 & \text{if } x \in B_i, \\ 0 & \text{else.} \end{cases} \tag{12}$$

Specifically, we choose the ordered set of basis functions to be in terms of a nested refinement of boxes $\{B_i\}$ covering M. Though Galerkin's and Ulam's methods are formally equivalent in the deterministic case, we are of the opinion that the Galerkin description is a more natural description in the stochastic setting.

3. Markov partitions of deterministic systems, and exact projection.

In this section, we discuss that a Markov partition is special for the FP operator of a deterministic dynamical system in that characteristic functions supported over those partition elements lead to an exact projection of the FP operator onto an operator of finite rank, a matrix.

For a one-dimensional transformation of the interval, the definition of a Markov partition [24] (see also [15, 25]) can be found in more recent references [3, 10, 18].

Definition. *A map of the interval $\tau : [a, b] \to [a, b]$ is* Markov *if there is a finite partition $\{I_j\}_{j=1}^N$ such that*

1. $\cup_{j=1}^N I_j = [a, b]$ *(covering property),*
2. $\mathrm{int}(I_j) \cap \mathrm{int}(I_k) = \emptyset$ *if* $j \neq k$ *(no overlap property),*
3. $\tau(I_j) = \cup_{i=1}^{q(j)} I_{k_i^{(j)}}$ *for some* $k_i^{(j)} \in \{1, 2, \ldots, N\}$, $i = 1, 2, \ldots, q(j)$ *(a grid interval maps completely across a union of intervals without "dangling ends" property).*

It is not hard to show that the set of characteristic functions forms a finite basis set of functions

$$\{\phi_j(x)\} = \{\mathbf{1}_{I_j}(x)\}_{j=1}^N, \tag{13}$$

such that Galerkin projection (10) is exactly onto an operator of finite rank or a matrix $A_{i,j}$. That is, (10) simplifies to

$$\begin{aligned} A_{i,j} = ([P_\tau \phi_i], \phi_j) &= \int_M [P_\tau \phi_i](x)\phi_j(x)dx, \\ &= \int_M \int_M \delta(x - \tau(y))\phi_i(y)\phi_j(x)dydx \\ &= \int_{I_j} \int_{I_i} \delta(x - \tau(y))dydx, \quad 1 \le i, j \le N. \end{aligned} \tag{14}$$

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If the map τ is in addition piecewise linear on its Markov partition, then $P_\tau[\phi_i(x)]$ is a linear combination of $\phi_j(x)$.

Similarly, there is a well-defined notion of an Anosov diffeomorphism with a Markov partition [3, 4, 9, 23], and so for such systems, it can be shown that characteristic functions supported over the corresponding Markov partition create a basis set such that (10) results in an operator of finite rank.

We take these observations as motivation for the following definition, which is meant to generalize the notion of a Markov partition to stochastic systems.

Definition. *Let $\{M, \mathcal{B}, \mu\}$ be a measure space and a transformation $\tau : M \to M$ be measurable. Then the transformation τ is* "basis Markov" *if there exists a finite set of basis functions $\{\phi_i(x)\}_{i=1}^N : M \to [0,1] \in L^1(M)$ such that the FP operator is operationally closed within Δ_N, where $\Delta_N = \mathrm{Span}(\{\phi_i(x)\}_{i=1}^N)$. That is, for any density $\rho \in \Delta_N$, its image $[P_\tau \rho](x)$ belongs to Δ_N.*

Remark 1. If a transformation τ is basis Markov, then, if we perform Galerkin's method, $A_{i,j} = (P_\tau[\phi_i], \phi_j)_{N\times N}$, with that basis set, it allows that, for any initial density which can be written as a linear combination of these basis functions,

$$\rho_0(x) = \sum_{i=1}^N c_i \phi_i(x), \tag{15}$$

or stated simply,

$$\rho_0(x) \in \Delta_N. \tag{16}$$

The action of the FP operator on such initial densities, $\rho_1(x) = P_\tau[\rho_0(x)]$, has the matrix presentation,

$$\mathbf{c}' = A \cdot \mathbf{c}, \text{ where } \rho_1(x) = \sum_{i=1}^N c_i' \phi_i(x), \tag{17}$$

and is well known as a linear operator from an N-dimensional vector space into itself. This emphasizes that the FP operator projects exactly to an operator of finite rank—a matrix.

Note that for a general finite set of functions, if we take a general linear combination of those functions and then apply the FP operator, we do not expect that the resulting density can be written as a (finite) linear combination of basis functions.

The following is a direct consequence of our definition of basis Markov in relationship to the usual definition of a Markov map, stating the sense in which basis Markov is a generalization.

Remark 2. Equation (14) implies that any piecewise linear Markov map, together with the characteristic functions supported over the partition elements, is basis Markov.

4. Basis Markov stochastic systems: A general case due to separable noise.

We analyze a dynamical system τ defined on an interval $M = [0, 1]$ with both ends identified and subjected to a specific form of the additive noise,

$$x' = \tau(x) + \xi. \tag{18}$$

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To specify the special case of the stochastic dynamical system written in (5), the stochastic perturbation will be characterized by the transition density $\mathcal{P}(x, y)$ of a transition from point x to y induced by noise. Describing the dynamics in terms of a probability density $\rho(x)$ its one-step evolution is governed by the stochastic FP operator,

$$\rho'(x) = [P_{\mathcal{P}}\rho](x) \ = \int_M \mathcal{P}\big(\tau(y), x\big)\, \rho(y)dy. \tag{19}$$

We will denote this stochastic FP operator by the symbol $P_{\mathcal{P}}$, referring to (6) in all that follows. The operator $P_{\mathcal{P}}$ acts on every probability density defined on M, and, in general, it cannot be represented by a finite matrix. However, in what follows we shall analyze a certain class of noise profiles for which such a representation is possible.

Definition. *The stochastic system of equations* (19) *is called* basis Markov *if there exists a finite set of basis functions* $\{\phi_i(x)\}_{i=1}^N : M \to [0, 1] \in L^1(M)$ *such that the FP operator* $P_{\mathcal{P}}$ *is operationally closed within* Δ_N*, where* $\Delta_N = \mathrm{Span}(\{\phi_i(x)\}_{i=1}^N)$.

We assume that the transition probability, $\mathcal{P}(x, y) \geq 0$, satisfies the following properties [21, 22]:

$$\begin{aligned} &\text{(a)} \quad \mathcal{P}(x, y) \equiv \mathcal{P}(x - y) = \mathcal{P}(\xi),\\ &\text{(b)} \quad \mathcal{P}(x, y) \equiv \mathcal{P}(x \bmod 1, y \bmod 1),\\ &\text{(c)} \quad \mathcal{P}(x, y) = \sum_{m,n=0}^{N} A_{mn}\, u_n(x)\, v_m(y) \end{aligned} \tag{20}$$

for $x, y \in \mathbb{R}$ and some finite N. Property (a) ensures that the distribution of the random variable ξ does not depend on the position x, while the periodicity condition is provided in (b). A noise profile fulfilling property (c) is called *separable* (decomposable), and it allows us to represent the dynamics of an arbitrary system with such a noise in a finite dimensional Hilbert space. Here $A = (A_{mn})_{m,n=0,\ldots,N}$ is a yet undetermined real matrix of expansion coefficients. Note that A characterizes the noise and does not depend on the deterministic dynamics τ. We assume that the functions u_n, $n = 0, \ldots, N$, and v_m, $m = 0, \ldots, N$, are continuous in $M = [0, 1)$ and linearly independent, and we can express $f \equiv 1$ as their linear combinations. Both sets of functions span bases in an $N+1$ Hilbert space. Their orthogonality is not required.

This term *separable noise* is concocted in analogy to *separable states* in quantum mechanics and *separable* probability distributions, since such a property was called $N+1$-separability by Tucci [26]. Making use of this crucial feature of the noise profile we may expand the kernel of the FP operator (19):

$$\rho'(y) = [P_{\mathcal{P}}\rho](y) = \int_0^1 \sum_{m,n=0}^{N} A_{mn} u_n(\tau(x)) v_m(y) \rho(x) dx \tag{21}$$

$$= \sum_{m,n=0}^{N} A_{mn} \left[\int_0^1 u_n(\tau(x))\rho(x)dx\right] v_m(y) \tag{22}$$

$$= \sum_{n=0}^{N} \left[\int_0^1 u_n(\tau(x))\rho(x)dx\right] \tilde{v}_n(y)$$

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for $y \in M$, where

$$\tilde{v}_n = \sum_{m=0}^{N} A_{mn} v_m. \tag{23}$$

Thus, any initial density is projected by the FP operator $P_{\mathcal{P}}$ into the vector space spanned by the functions $\tilde{v}_m$, $m = 0, \ldots, N$.

Assuming that a given density $\rho(x)$ belongs to this space, we can expand it in this basis,

$$\rho(x) = \sum_{m=0}^{N} q_m \, \tilde{v}_m(x). \tag{24}$$

Expanding ρ' in an analogous way, we will describe it by the vector $\vec{q}\,' = \{q'_0, \ldots, q'_N\}$.

Let B denote a matrix of integrals,

$$B_{nm} = \int_0^1 u_n(\tau(x)) v_m(x) dx, \tag{25}$$

where $n, m = 0, \ldots, N$. Observe that B depends directly on the system τ and on the noise via the basis functions u and v. Making use of this matrix, the one-step dynamics (23) may be rewritten in a matrix form

$$q'_n = \sum_{m=0}^{N} D_{nm} \, q_m, \quad \text{where } D = BA \tag{26}$$

and A is implied by (20). In this way we have arrived at a representation of the FP operator $P_{\mathcal{P}}$ by a matrix D of size $(N+1) \times (N+1)$, the elements of which read

$$D_{nm} = \int_0^1 u_n(\tau(x)) \tilde{v}_m(x) dx, \quad n, m = 0, \ldots, N. \tag{27}$$

With (26), we now see that random dynamical systems with noise satisfying condition (20) allow a finite dimensional subspace which is preserved.

Although the probability is conserved under the action of $P_{\mathcal{P}}$, the matrix D need not be stochastic. This is due to the fact that the functions $\{\tilde{v}_m(x)\}$ forming the expansion basis in (24) were not normalized. We shall then compute their norms,

$$s_m = \int_0^1 \tilde{v}_m(y) dy = \sum_{n=0}^{N} A_{mn} b_n, \tag{28}$$

where

$$b_n = \int_0^1 v_n(y) dy. \tag{29}$$

Let $K \leq N+1$ denote the number of nonzero components of the vector $\vec{s}$, and let $k = 1, \ldots, K$ runs over all indexes $n \in 0, \ldots, N+1$, for which $s_k \neq 0$. Then the rescaled vectors,

$$V_k(y) := \tilde{v}_k(y)/s_k, \tag{30}$$

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are normalized:

$$\int_0^1 V_k(y)dy = 1. \tag{31}$$

The normalization condition $\int_0^1 \rho(x)dx = 1$ implies

$$\int_0^1 \sum_{m=0}^{N} q_l \tilde{v}_m(x)dx = \sum_{m=0}^{N} q_m s_m = \sum_{k=1}^{K} q_k s_k = 1. \tag{32}$$

The same is true for the transformed density,

$$\sum_k q'_k s_k = 1. \tag{33}$$

Hence this scalar product is preserved during the time evolution. Making use of the rescaled coefficients

$$c_k := q_k s_k, \tag{34}$$

the dynamics (26) reads

$$c'_k = q'_k s_k = \sum_j D_{kj}\, q_j s_k = \sum_j D_{kj}\, \frac{s_k}{s_j}\, q_j s_j =: \sum_j T_{kj}\, c_j. \tag{35}$$

By construction the coefficients c_k sum to unity. Since some of them can in general be negative, the transition matrix

$$T_{kj} \;\equiv\; D_{kj}\, \frac{s_k}{s_j} = \sum_{ii'} D_{kj}\, \frac{A_{ki} s_i}{A_{ji'} s_{i'}} \tag{36}$$

need not be stochastic. In the above equation, all indices run from 1 to K and the coefficients s_k are nonzero by construction.

It is interesting to distinguish a special class of noises for which all functions corresponding to nonzero values of the components s_k are nonnegative: $\tilde{v}_k(x) \geq 0$ for $x \in [0,1]$ and $k = 1,\ldots,K$. This implies that for any probability density ρ its expansion coefficients q_k in (24) are not negative. Furthermore, the normalization constants of $\tilde{v}_k$ are nonnegative, $s_k > 0$, $k = 1,\ldots K$, and so are the coefficients c_k and c'_k given in (34), (35). Hence vectors c and c' form normalized K-point probability distributions, and so in this case the transition matrix T of size K is stochastic. The dimensionality $K \leq N+1$ is determined by the parameter N and the choice of the basis functions $\{v_l(x)\}$ entering (20).

5. A special case: Cosine noise. We will now discuss a particularly simple case of the separable noise described above and introduced in [21]. Let

$$\mathcal{P}_N(\xi) = \mathrm{C_N} \cos^N(\pi\xi), \tag{37}$$

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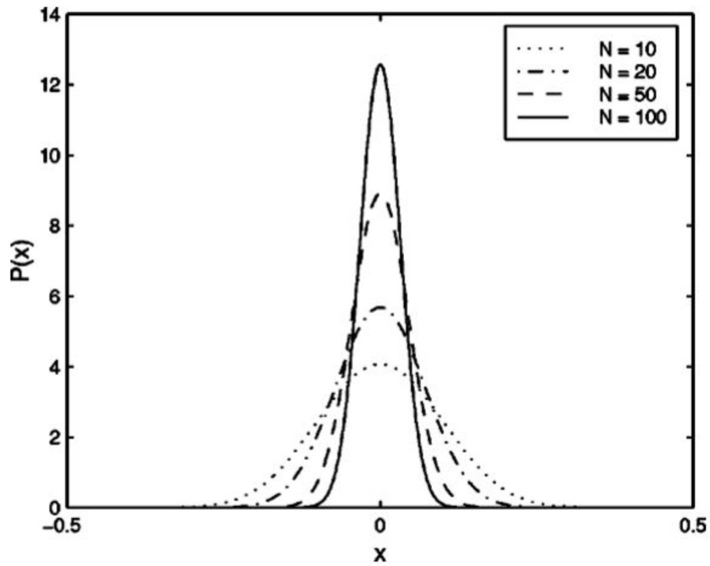

Figure 1. *The cosine noise of (37) closely resembles a normal noise profile, but with finite support. Several values of N are shown, with decreasing standard deviation with increasing N.*

where N is even ($N = 0, 2, \ldots$), and with the normalization constant

$$\mathrm{C_N} = \sqrt{\pi}\,\frac{\Gamma[N/2+1]}{\Gamma[(N+1)/2]}. \tag{38}$$

See Figure 1, in which we can see the decreasing standard deviation with respect to increasing N. This type of noise reminds us of a normal distribution, but of compact support.

The parameter N controls the strength of the noise measured by its variance

$$\sigma^2 = \frac{1}{2\pi^2}\Psi'\left(\frac{N}{2}+1\right) = \frac{1}{12} - \frac{1}{2\pi^2}\sum_{m=1}^{N/1}\frac{1}{m^2}, \tag{39}$$

where Ψ' stands for the derivative of the digamma function.

For the expansion (20) we use basis functions

$$\begin{aligned} u_m(x) &= \cos^m(\pi x)\sin^{N-m}(\pi x),\\ v_n(y) &= \cos^n(\pi y)\sin^{N-n}(\pi y), \end{aligned} \tag{40}$$

where $x \in M$ and $m, n = 0, \ldots, N$. Expanding the cosine as a sum to the Nth power in (37), we find that the $(N+1)\times(N+1)$ matrix A defined by (20) is diagonal:

$$A_{mn} = a_m\delta_{mn}, \text{ with } a_m = \mathrm{C_N}\binom{N}{m}. \tag{41}$$

Integrating trigonometric functions, we find the coefficients

$$b_m = \int_0^1 \cos^m(\pi x)\sin^{N-m}(\pi x)dx = \frac{2}{\pi N}\frac{\Gamma[(m+1)/2]\,\Gamma[(N-m+1)/2]}{\Gamma(N/2)} \tag{42}$$

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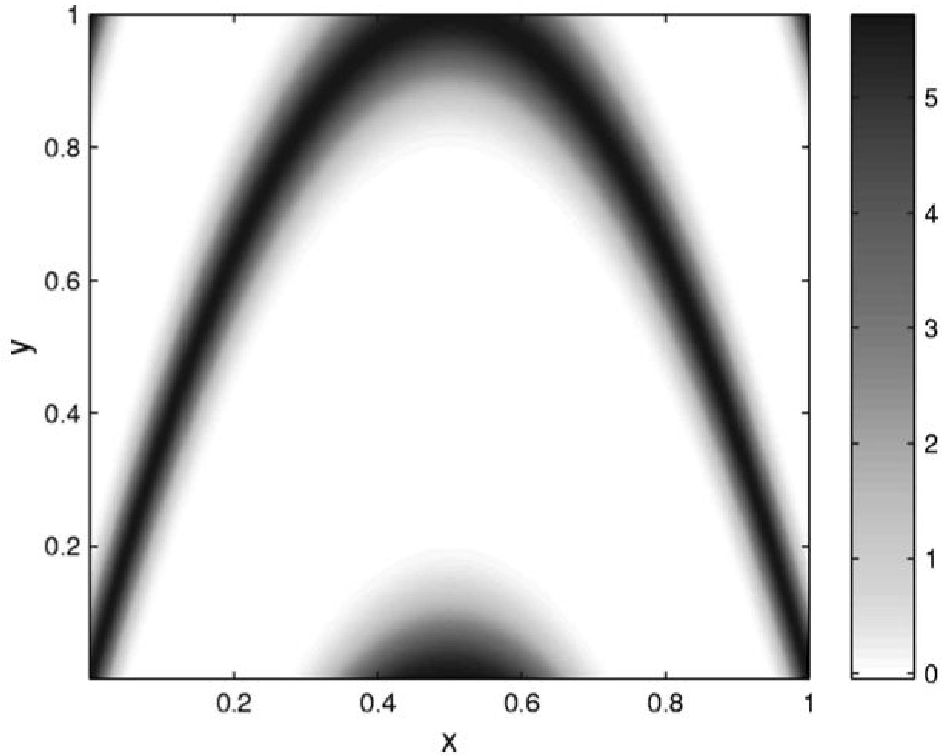

Figure 2. *The transition kernel $\mathcal{P}_N(f(x), y)$ for the logistic map $\tau(x) = 4x(1-x)$, with $N = 20$ and with cosine noise due to $N = 20$; compare to Figure 1. Note the periodicity of x of period 1.*

and

$$s_m = a_m b_m, \tag{43}$$

which are nonzero only for even values of m. Hence the size $K \times K$ of the transition matrix reads

$$K = N/2 + 1, \tag{44}$$

and the expression (36) takes the form

$$T_{kj} = D_{mn} \frac{a_m b_m}{a_n b_n}, \quad \text{where } k, j = 1, \ldots, K, \; m = 2(k-1), \; n = 2(j-1). \tag{45}$$

For the noise (37) discussed here all functions $\tilde{v}_m$ for even m, which contribute to the matrix T, are nonnegative; hence, as discussed in the previous section, the transition matrix T is stochastic. We find in this case that the transition kernel reminds us of a fuzzy but periodically repeated version of the map. See Figure 2. However, the FP operator embeds to a transition matrix T, which "appears" roughly as a different form of the original map; see Figure 3.

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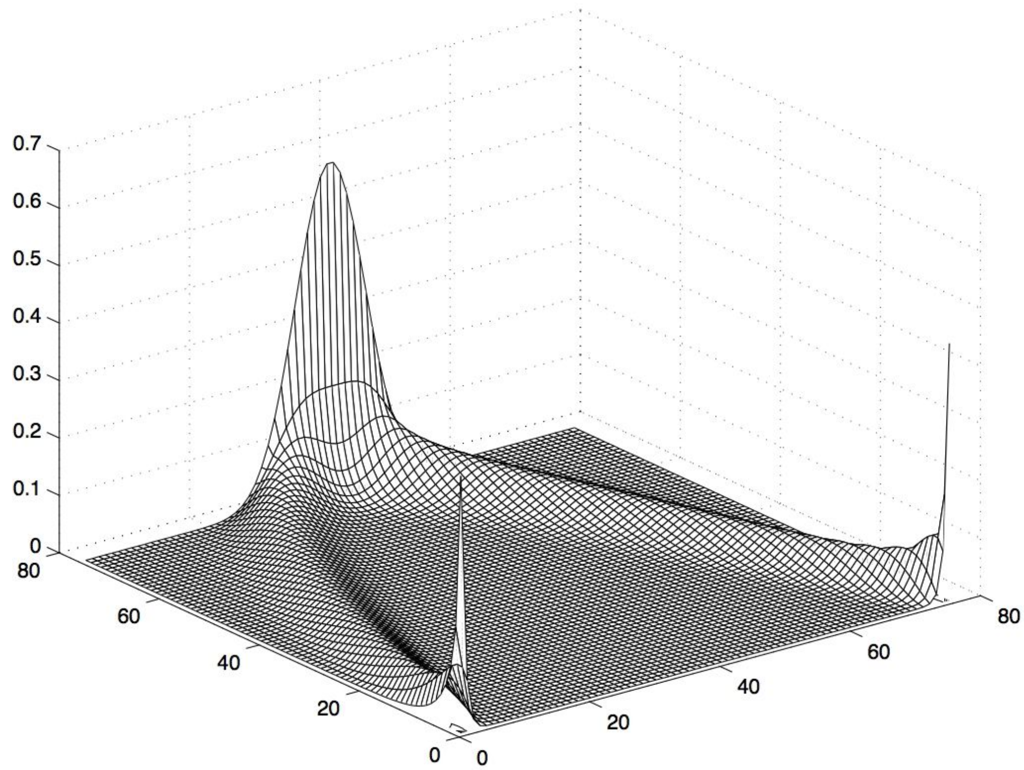

Figure 3. *The stochastic matrix T_{150} shown, from (36), exactly represents the stochastic FP operator of the stochastic tent map (47) with trig noise (37) and basis set (40) using $N = 150$. Note that $T^{(150)}$ is a matrix of size $N/2 + 1 = 150/2 + 1 = 76$ square. Compare to the matrices in (48) of smaller N.*

There is an interesting correspondence between the spectra of eigenvalues of the two matrices D and T. Since T is stochastic, its largest eigenvalue is equal to unity. Moreover, it is the only eigenvalue with modulus one, which follows from the fact that the kernel $\mathcal{P}(x, y)$ vanishes only for $x - y = 1/2 \pmod 1$, and the two-step probability function is everywhere positive:

$$\int_M \mathcal{P}(x, z)\mathcal{P}(z, y)dz > 0 \text{ for } x, y \in M \tag{46}$$

(see [16, Th. 5.7.4]). A particularly useful consequence and simplification is that the eigenstate corresponding to the largest eigenvalue of the matrix represents the invariant density of the system, $\rho_* = P_f(\rho_*)$; this can be easily found numerically by diagonalizing T.

All of the other eigenvalues are included inside the unit circle and their moduli $|\lambda_i|$ characterize the decay rates. It is worth emphasizing that the spectra of both matrix representations of the FP operator, by matrices D of size $(N + 1) \times (N + 1)$ used in [20, 21, 22] and the stochastic T matrices of size $(N/2+1) \times (N/2+1)$ developed here, coincide up to the additional $N/2$ eigenvalues which are equal to zero; see the appendix for details.

For concreteness let us discuss an exemplary one-dimensional dynamical system, a tent map:

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$$\tau(x) := \begin{cases} 2x & \text{if} \quad 0 \leq x \leq 1/2, \\ 2(1-x) & \text{if} \quad 1/2 \leq x \leq 1. \end{cases} \tag{47}$$

Simple integration allows us to obtain the analytic form of the transition matrix $T^{(N)}$ for the tent map (47) perturbed by additive noise characterized by small values of N,

(48)

$$T^{(2)} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad T^{(4)} = \frac{1}{24}\begin{bmatrix} 11 & 3 & 11 \\ 6 & 6 & 6 \\ 7 & 15 & 7 \end{bmatrix}, \quad T^{(6)} = \frac{1}{320}\begin{bmatrix} 145 & 25 & 25 & 145 \\ 69 & 45 & 45 & 69 \\ 51 & 75 & 75 & 51 \\ 55 & 175 & 175 & 55 \end{bmatrix}.$$

In the simplest case $N = 2$ the transition matrix is bistochastic, but it is not so for larger N. However, for this system, the matrix $T^{(N)}$ is of rank one for arbitrary value of the noise parameter N. The spectrum of T contains one eigenvalue equal to unity and all others equal to zero. This implies that every initial density is projected onto an invariant density already after the first iteration of the map. This is not the case for other dynamical systems τ, including the logistic map $\tau_r(x) = rx(1-x)$, for which the spectrum contains several resonances—eigenvalues of moduli smaller than one—which describe the decaying modes of the system [21].

In this way we have established a relation between a sequence of noisy systems τ_N and the deterministic dynamical system τ. A stochastic system (18) with the noise profile (37) for a fixed noise parameter N is described by a stochastic matrix $T^{(N)}$ of size $K = N/2 + 1$ and acts in the Hilbert space $\mathcal{H}_K$.

We have shown that the sequence of transition matrices $T^{(N)}$ *corresponds* to the dynamical system τ in the sense that the sequence μ_N of the invariant measures of $T^{(N)}$ converges weakly to the τ-invariant measure μ in the deterministic limit $N \to \infty$. Furthermore, for any initial density ρ the sequence of vectors ρ'_N transformed by $P_{\mathcal{P}_N}$ converges weakly to the density transformed by the FP operator associated with τ. Observe that the above property holds not only for one-dimensional systems but also for dynamical system τ in higher dimensional measure spaces.

6. General example. In this section we construct a fairly rich family of transition densities satisfying the assumptions (20). Let $\{g_N\}_{N\geq 1}$ be a sequence of C^2 (this condition can be weakened) nonnegative functions with support in $[-1/2, 1/2]$ such that $g_N(-1/2) = g_N(1/2)$ for all $N \geq 1$ and which converges to Dirac's delta δ_0 as $N \to \infty$.

Each g_N, which can be also seen as a 1-periodic function on the whole real line, can be approximated by its partial Fourier sum arbitrarily close in the supremum norm. Let

$$h_N(\xi) = c_{S(N)} + a_{0,N} + 2\sum_{s=1}^{S(N)} \left(a_{s,N}\cos(2s\pi\xi) + b_{s,N}\sin(2s\pi\xi)\right) \tag{49}$$

be an approximation obtained from Fourier approximation by shifting it up by a small constant $c_{S(N)}$ to ensure $h_N \geq 0$ on $[-1/2, 1/2]$. We have $c_{S(N)} \to 0$ as $S(N) \to \infty$. We can make the

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functions h_N also converge to Dirac's delta δ_0 as $N \to \infty$. Using the functions h_N we define a family of densities:

$$\mathcal{P}_N(\xi) = h_N(\xi) / \int_{-1/2}^{1/2} h_N(t)dt, \quad N = 1, 2, 3, \dots, \tag{50}$$

and then a family of transition densities

$$\mathcal{P}_N(x, y) = \mathcal{P}_N(x - y), \quad N = 1, 2, 3, \dots.$$

Since

$$\cos(2s\pi(x - y)) = \cos(2s\pi x) \cos(2s\pi y) + \sin(2s\pi x) \sin(2s\pi y),$$
$$\sin(2s\pi(x - y)) = \sin(2s\pi x) \cos(2s\pi y) - \cos(2s\pi x) \sin(2s\pi y),$$

it is clear that the assumptions (20)(c) are satisfied with $u_n(x)$ equal to $\cos(2s\pi x)$ or $\sin(2s\pi x)$ and $v_m(y)$ equal to $\cos(2s\pi y)$ or $\sin(2s\pi y)$ for $0 \le s \le S(N)$. It is also clear that for each $x \in [0, 1]$, $\mathcal{P}_N(x, \cdot)$ converges to Dirac's delta δ_x as $N \to \infty$. To have the condition (3) of the next section satisfied it is enough to start with even functions g_N.

Example. Let $g(\xi) = (0.2 + x^2) \exp(-x^2)$ and $g_N(\xi) = Ng(N\xi)$, restricted to $[-1/2, 1/2]$ and extended periodically to the whole real line, $N \ge 1$. Then, the g_N's are positive and converge to Dirac's δ_0 as $N \to \infty$. In particular, let us consider g_6. Its Fourier approximation, with $S(6) = 5$, is

$$1.24032 + 1.14838 \cos(2\pi\xi) - 0.470309 \cos(4\pi\xi) - 0.530699 \cos(6\pi\xi)$$
$$- 0.163161 \cos(8\pi\xi) - 0.0225748 \cos(10\pi\xi).$$

We have used such a poor approximation to make the example simpler. We can choose constant $c_{S(6)} = 0$ and after normalization we obtain

$$\mathcal{P}_6(\xi) = 1 + 0.92587 \cos(2\pi\xi) - 0.37918 \cos(4\pi\xi) - 0.42787 \cos(6\pi\xi)$$
$$- 0.131547 \cos(8\pi\xi) - 0.01820 \cos(10\pi\xi).$$

See the transition density in Figure 4.

We have

$$\begin{aligned}\mathcal{P}_6(x - y) = 1 &+ 0.92587 \cos(2\pi x) \cos(2\pi y) + 0.92587 \sin(2\pi x) \sin(2\pi y) \\ &- 0.37918 \cos(4\pi x) \cos(4\pi y) - 0.37918 \sin(4\pi x) \sin(4\pi y) \\ &- 0.42787 \cos(6\pi x) \cos(6\pi y) - 0.42787 \sin(6\pi x) \sin(6\pi y) \\ &- 0.131547 \cos(8\pi x) \cos(8\pi y) - 0.131547 \sin(8\pi x) \sin(8\pi y) \\ &- 0.01820 \cos(10\pi x) \cos(10\pi y) - 0.01820 \sin(10\pi x) \sin(10\pi y).\end{aligned}$$

See the transition kernel in Figure 5.

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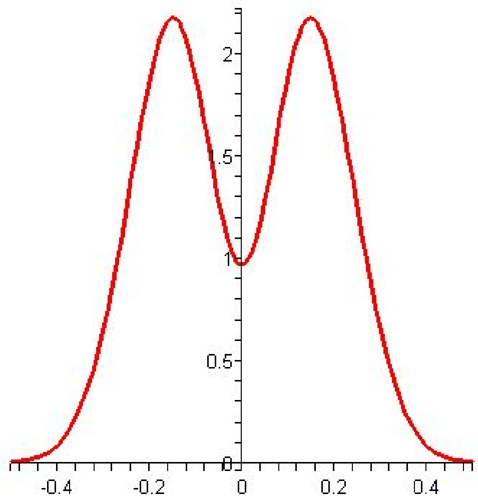

Figure 4. *The transition density $P_6(\xi)$.*

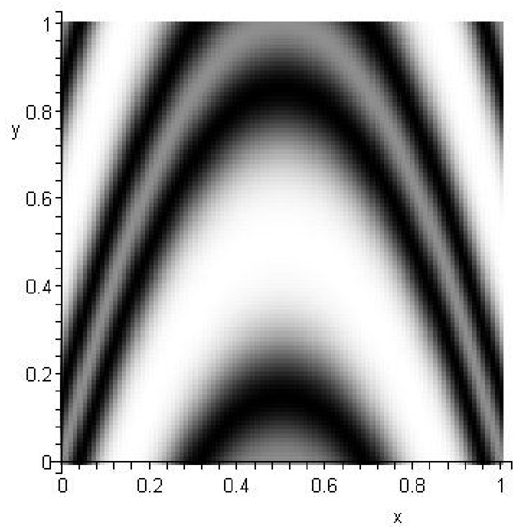

Figure 5. *The transition kernel $\mathcal{P}_6(\tau(x), y)$ for the logistic map $\tau(x) = 4x(1-x)$, with $S(6) = 5$.*

In the notation of section 4 let us define

$$\begin{aligned} u_0(x) &= 1, \\ u_{2s+1}(x) &= \cos(2(s+1)s\pi x), \quad s = 0, 1, 2, 3, 4, \\ u_{2s}(x) &= \sin(2s\pi x), \quad s = 1, 2, 3, 4, 5, \\ v_0(y) &= 1, \\ v_{2s+1}(y) &= \cos(2(s+1)\pi y), \quad s = 0, 1, 2, 3, 4, \\ v_{2s}(y) &= \sin(2s\pi y), \quad s = 1, 2, 3, 4, 5. \end{aligned}$$

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Then, matrix $A = (A_{mn})_{0 \le m,n \le 10}$ is the diagonal matrix with the diagonal

$$[1, 0.92587, 0.92587, -0.37918, -0.37918, -0.42787, -0.42787, -0.131547, -0.131547, -0.01820, -0.01820],$$

and we have

$$\tilde{v}_m = A_{mm} v_m, \quad m = 0, 1, \ldots, 10.$$

Let us consider the dynamics given by the logistic map $\tau : x \mapsto 4x(1-x)$. Matrix D defined in (27) and representing FP operator $P_{\mathcal{P}_6}$ is

(51)

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2441 & -0.3096 & 0 & -0.08859 & 0 & -0.1209 & 0 & -0.01639 & 0 & -0.0007850 & 0 \\ -0.1717 & 0.2016 & 0 & -0.1313 & 0 & 0.00547 & 0 & 0.01382 & 0 & 0.001494 & 0 \\ 0.1752 & -0.1940 & 0 & -0.09397 & 0 & 0.06079 & 0 & 0.02395 & 0 & -0.002347 & 0 \\ -0.1372 & 0.1658 & 0 & -0.0159 & 0 & 0.06947 & 0 & -0.02997 & 0 & -0.004869 & 0 \\ 0.1436 & -0.1506 & 0 & -0.07445 & 0 & 0.08111 & 0 & -0.006781 & 0 & -0.002915 & 0 \\ -0.1178 & 0.1397 & 0 & 0.00971 & 0 & 0.02136 & 0 & -0.02983 & 0 & 0.001684 & 0 \\ 0.1246 & -0.1268 & 0 & -0.06176 & 0 & 0.07547 & 0 & -0.01636 & 0 & -0.0003029 & 0 \\ -0.1051 & 0.1223 & 0 & 0.01799 & 0 & -0.00139 & 0 & -0.01922 & 0 & 0.002802 & 0 \\ 0.1116 & -0.1116 & 0 & -0.05331 & 0 & 0.06756 & 0 & -0.01849 & 0 & 0.001108 & 0 \\ -0.09589 & 0.1098 & 0 & 0.02112 & 0 & -0.01226 & 0 & -0.01162 & 0 & 0.002350 & 0 \end{bmatrix}.$$

The eigenvalues of D are

$$1, -.4086428809, 0.0800412, -0.0117582, 0.00331119, -0.000739709, 0, 0, 0, 0, 0.$$

Although in this case all eigenvalues of D are real, in general they are complex. Since matrix D is real, the eigenvalues are placed symmetrically with respect to the real axis. The eigenvector for eigenvalue 1 is

$$w = [1, 0.164834, -0.154604, 0.139482, -0.107445, 0.116956, -0.0938748, 0.102202, -0.0843296, 0.0918404, -0.0772548],$$

and it provides a rough approximation $\sum_{m=0}^{10} w[m]\tilde{v}_m(\xi)$ to the τ-invariant density. A much better approximation shown in Figure 6 is obtained by taking the same noise profile for $N = 40$ and $S = 30$, which results in matrix D of size $2S + 1 = 61$.

For a comparison we performed Ulam's approximation of the invariant density of τ using an $N \times N$ matrix with $N = 61$. For $\phi_m = N \cdot \mathbf{1}_{[(m-1)/N, m/N]}$, Ulam's probabilistic matrix $U = \{U_{ij}\}$ can be obtained by putting

$$U_{ij} = (1/N) \int_0^1 \phi_i(t)\phi_j(\tau(t))dt, \quad 1 \le i, j \le N.$$

We found the 1-eigenvector w of U, and the function $f_N = \sum_{m=1}^{N} w[m]\phi_m$ is Ulam's approximation to τ-invariant density. It is shown in Figure 7.

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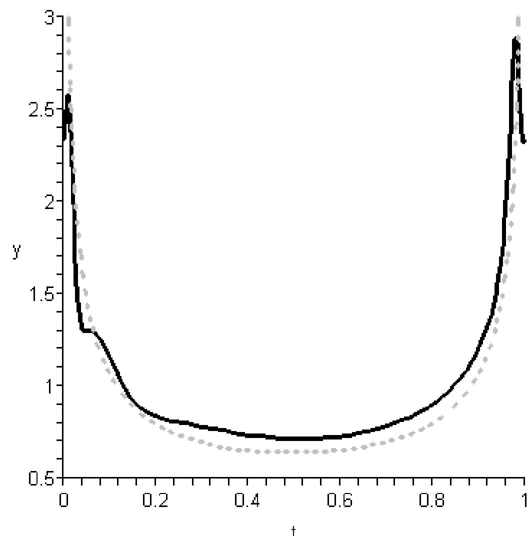

Figure 6. *An approximation to the invariant density of the logistic map (dashed line) obtained as an invariant density of transition matrix D of size 61×61 (solid line).*

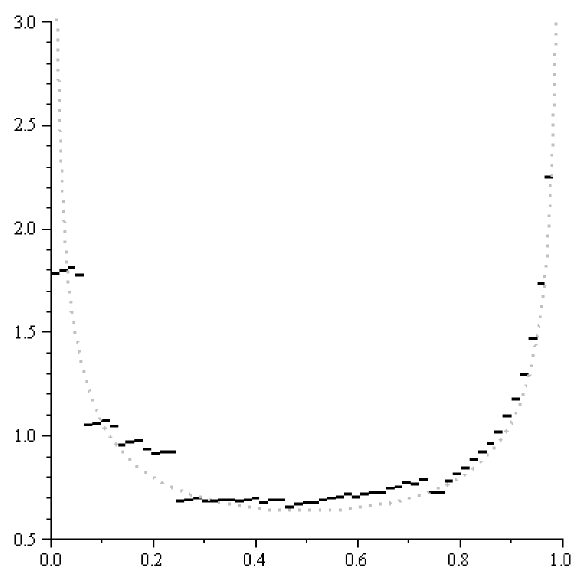

Figure 7. *An approximation to the invariant density of the logistic map (dashed line) obtained by Ulam's method with 61×61 matrix (solid line).*

The L^1 errors of approximation were comparable: 0.17 for Ulam's method and 0.20 for our method. Our method produces a smooth approximating function which is nicer for a smooth invariant density. Our method is also more general in the sense that it can be used to approximate not only the invariant density itself but also the invariant density of a random

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perturbation of a map by a possibly very strange perturbing distribution. This was shown in the example above. On the other hand, Ulam's method is definitely simpler and its theoretical properties are well studied.

7. Approximation by basis Markov maps.

While not all maps and noise profiles allow for the map to be basis Markov, in this section we will show that a non–basis Markov map may be weakly well approximated by basis Markov maps. In this sense, the finite approximations offered by basis Markov maps can be thought of as a good description of the general behavior, since the invariant measures of the finite approximations due to the basis Markov maps have weak-* limits to the invariant measures of the general maps.

Let us consider a family of the transition probabilities $\mathcal{P}_N(\cdot,\cdot)$ such that, for each $x \in M$, $\mathcal{P}_N(x,\cdot)$ converges to Dirac's delta δ_x as $N \to \infty$.

We require the following assumptions about the transition probabilities $\mathcal{P}_N(\cdot,\cdot)$:

1. $\mathcal{P}_N(\cdot,\cdot)$ is measurable as a function of two variables.
2. For every x we have $\int_M \mathcal{P}_N(x,y)dy = 1$.
3. For every $y \in M$ we have
$$\int_M \mathcal{P}_N(x,y)dx = 1.$$
4. Let $B(x,r) = \{y : |x-y| < r\}$ and
$$p_N(x,r) = \int_{M\setminus B(x,r)} \mathcal{P}_N(x,y)dy. \tag{52}$$
Then, for any $r > 0$,
$$p_N(r) = \sup_{x\in M} p_N(x,r) \to 0 \quad \text{as} \quad N \to +\infty.$$

Assumptions 1–3 are typical for probability measures, while assumption 4 is also rather mild, and it is easy to check that all four assumptions are satisfied by the cosine noise (37).

Under these assumptions, the following can be easily proved.

Proposition. *Let $M = [0,1]$. For any $\rho \in L^1(M)$ we have*
$$\int_M \rho(x)\mathcal{P}_N(x,y)dx \to \rho(y) \text{ as } N \to \infty \tag{53}$$
in $L^1(M)$.

In Theorem 1 below we assume that the transformation $\tau : [0,1] \to [0,1]$ is continuous. This assumption can be weakened (say, to piecewise continuous) if we impose additional restrictions on the transition probabilities $\mathcal{P}_N$ (say, such that all measures μ_N and their weak limits are continuous measures; see, for example, [8]).

Theorem 1. *Let the transformation τ be continuous. Under the assumptions 1, 2, and 4, it follows that if μ_N is an invariant measure of the stochastic perturbation of transformation f defined by the transition probability $\mathcal{P}_N$, then every weak-$*$ limit point of the set $\{\mu_N : N \geq 1\}$ is an f-invariant measure.*

This theorem can be proved following the ideas of Khasminskii [12].

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A more precise result can be proved under more restrictive assumptions on the transformation τ.

Theorem 2. *Let the transformation τ be piecewise C^2 and piecewise expanding, i.e., $|\tau'| > 2$, where it exists. Then, under the assumptions 1–4, every weak-$*$ limit point of the set $\{\mu_N : N \geq 1\}$ is a τ-invariant absolutely continuous measure.*

This result was proved in Theorem I.B. of [8]. The perturbations we consider are of "convolution type" and since we treat an interval as a circle an extra factor of 2 does not occur. The example of the famous W-map [11] shows that the condition $|\tau'| > 2$ cannot be weakened.

8. Concluding remarks. In this work we have introduced the concept of basis Markov stochastic systems, for which the associated FP operator is finite. This property resembles the class of deterministic systems with a Markov partition. However, the Markov partition is characteristic to a very special class of deterministic systems, while the basis Markov property is related to the kind of stochastic perturbation. It holds for any deterministic system τ, subjected to an additive noise with a profile satisfying the separability condition (20). In this way such a random dynamical system can be described by a stochastic transition matrix of a finite size K, which diverges in the deterministic limit.

We have shown an intimate relationship between the sequence of stochastic matrices which act in the space of K-point probability distributions and the FP operator P_τ of the deterministic system, which acts in the infinite dimensional space: In the deterministic limit $K \to \infty$ the invariant densities of stochastic matrices converge in a weak sense to the invariant measure of the deterministic system τ. Thus, constructing the transition matrices T and decreasing the noise strength (and increasing the dimensionality K), one may construct arbitrary approximations of the FP operator P_τ.

Some discussion regarding generality is in order. While it is not clear at this stage how many families of examples exist that satisfy the properties in (20), we presented a concrete example in section 5, the cosine noise example, (37), with corresponding basis functions (40). We find this example instructive due to its general appearance as similar to the familiar Gaussian distribution and the fact that it provides a finite representation of the FP operator P_τ by a stochastic transition matrix T. Furthermore, in section 6 we presented a general technique of designing one-dimensional noise profiles which satisfy the separability conditions (20).

Note that the described method is not restricted to one-dimensional systems. On the contrary, the entire construction can be directly applied to a general case of multidimensional dynamical systems. In particular, the definition (20)(c) of separable noise profiles works for the case of an L-dimensional system, provided the variables x and y represent vectors with L components each.

If the dynamical system acts on the L-torus, for example, $M = [0, 1]^L$, one can take the Cartesian product of the cosine noise (37) setting

$$\mathcal{P}_N(\xi_1, \ldots, \xi_L) = C_N^L \cos^N(\pi\xi_1) \cos^N(\pi\xi_2) \cdots \cos^N(\pi\xi_L), \tag{54}$$

where $\xi_k = x_k - y_k$ and $k = 1, \ldots, L$. This form of the additive noise was used in [20] to analyze a two-dimensional system (a variant of the baker map) and to compare the spectral properties of the FP operator associated with the classical stochastic system with properties

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of the propagator of the corresponding quantum evolution. In such a case the deterministic limit of the classical noisy system, $K \to \infty$, is related to the classical limit, $\hbar \to 0$, of the corresponding quantum dynamics.

Note that for basis Markov stochastic systems, the transition matrices T *exactly* describe the action of the dynamical system with additive noise on densities. Thus our construction differs from an approach applied in [13, 19, 28], where a finite dimensional description of the density dynamics of a deterministic system was achieved by truncation of an infinite transition operator P_τ to the finite dimension K. The effect of such a truncation may also be regarded as a kind of noise depending on the matrix size K and the base, in which P_τ is represented. On the other hand, in our case a suitable choice of the noise profile added to the deterministic system distinguishes a relevant basis, in which the FP operator of the perturbed system is finite.

Appendix. Isospectral matrices. In this appendix we show that the matrix D defined by (27) and used in [20, 21, 22] to represent the FP operator and the stochastic transition matrix T share the same nonzero part of the spectrum. We make use of the following algebraic result.

Lemma. *Let A be a square matrix of size $N \times N$ and $\vec{s}$ a vector of length N containing only nonzero entries. Then the matrix*

$$B_{jk} \equiv A_{jk} \frac{s_j}{s_k} \tag{55}$$

has the same spectrum as A.

(There is no summation over repeating indices.)

Proof. To study equation $\det(B - \lambda \mathbb{1}) = 0$ we start analyzing an exemplary term P^B of the determinant. It consists of a product of N elements $B_{i,\sigma(j)}$, where $\sigma(i)$ stands for a certain permutation of the indices. The product of N factors of the type $s_i/s_{\sigma(i)}$ is equal to unity so that

$$P_\sigma^B = \prod_i B_{i,\sigma(i)} = \prod_i B_{i,\sigma(i)} \frac{s_1 s_2 \cdots s_N}{s_1 s_2 \cdots s_N} = \prod_i A_{i,\sigma(i)}. \tag{56}$$

Thus every term contributing to the free coefficient of the characteristic equation will be the same, $P_\sigma^B = P_\sigma^A$; hence these coefficients for both matrices A and B are equal. Since the diagonal elements of both matrices coincide, $B_{jj} = A_{jj}$, all terms forming the coefficients standing by an arbitrary power of λ are the same for both matrices. Therefore, characteristic equations for both matrices are equal and so are their spectra. ■

Treating all nonzero elements of the vector s_k, $k = 1, \dots, K$, as vector $\vec{s}$, we may apply the lemma to (36) and obtain equivalence of the spectrum of T and the nonzero part of the spectrum of D. Since integrals (25) vanish for odd values of m, every second column of D is equal to zero, and the remaining $N/2$ eigenvalues of D are equal to zero.

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