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UNLOC: Optimal unfolding localization from noisy distance data

Jie Sun
Department of Mathematics, Department of Physics, Department of Computer Science, Clarkson Center for Complex Systems Science (C^3S^2),
Clarkson University, Potsdam, NY 13699.
sunj@clarkson.edu

Tianqi Yang, Kevin V. Mack, Mahesh K. Banavar
Department of Electrical and Computer Engineering, Clarkson Center for Complex Systems Science (C^3S^2),
Clarkson University, Potsdam, NY 13699.
{tiyang, mackkv, mbanavar}@clarkson.edu

Abstract. Target localization is an important problem in signal processing and sensor networks, with many application areas including security, consumer electronics, and health monitoring. In this paper, we formulate target localization as an inverse problem: given the locations of a set of anchors and noisy distance measures to a target, the localization problem is to estimate the unknown location of the target. We propose to solve the localization problem using an unfolding-based optimization. We show that the corresponding stress optimization, despite being a nonlinear problem, yields a global optimum that can be approximated using an efficient iterative algorithm. We term our computational approach as UNLOC, unfolding-based localization, and benchmark its effectiveness on both synthetic data and lab-generated experimental data. The proposed localization technique generally produces accurate target location, and the quality of localization can be further improved by an appropriate choice of weights in the objective function of optimization.

Key words and phrases: localization, unfolding, quadratic optimization

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1. Introduction

Wireless sensor networks (WSNs) have been commonly employed in many applications including localization and detection in control systems, and environmental protection [26]. In applications such as water monitoring or event detection, accurate knowledge of the locations of events is critical. Localization algorithms are used in conjunction with sensors to estimate the locations of targets. In a typical localization problem, the sensors are divided into two types (see Figure 1): (1) anchors at known locations; and (2) target nodes at unknown locations.

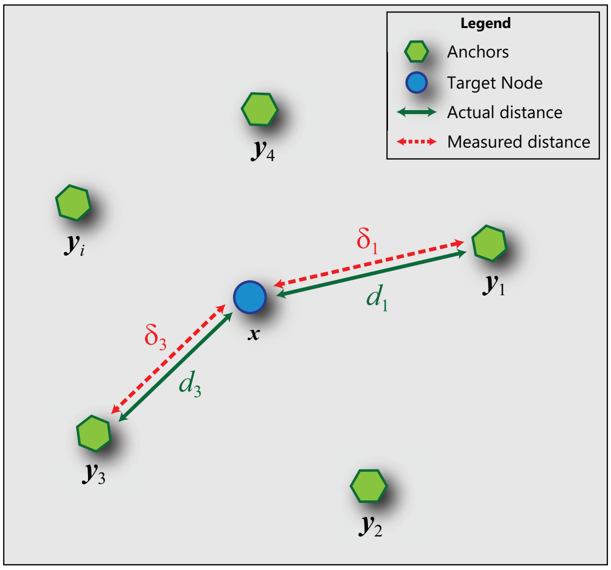

Figure 1. Schematic illustration of the sensor field used for localization. There are two types of sensors: the anchors are depicted by (green) hexagons and the target node is shown as the (blue) circle. The cross distance between anchor i and the target node is denoted by d_i, and is measured (subject to noise) to produce δ_i. Using the location of anchors $\boldsymbol{y}_i$ and the noisy distances δ_i, the goal is to estimate the location of the target node, $\boldsymbol{x}$.

In general, localization techniques can be classified into range-based and direction-based approaches [16, 19]. Common range-based approaches are time of arrival (TOA), time difference of arrival (TDOA), and received signal strength (RSS). Direction-based approaches include direction of arrival (DOA) estimation techniques implemented by employing antenna arrays at each sensor [9, 16, 26]. Hybrid methods such as large aperture array (LAA) localization [14] are also sometimes used.

In range-based localization, distance estimates between the target and each anchor are first obtained, and then used to estimate the location of the target. The localization algorithms estimate the location of the target by minimizing the error between the actual distance between the target and the anchors, and the respective measured distances. Methods to solve the problem are predominantly least-squares based [14, 25–27], with differences mainly in the individual terms in the summation of the squared model error [4, 8]. For example, in [4], three distinct least-squares problems (all of which are non-convex) are formulated and (partially) solved to provide localization estimates. The results are compared under different noise assumptions and measurement conditions.

Therein, the optimization takes equal contributions from the measured and model-fitted anchor-to-target distance, effectively treating noise as identically distributed across different measurements. However, in most applications, the measurement noise is related to the measured distance in some way and therefore, distance measurements should not be equally weighed when attempting to estimate the target location [6].

In this paper, we consider the general problem where the noise in measurement can be from any arbitrary model. We formulate a general class of forward models and show that the target localization problem can be treated as an inverse problem. In order to solve the inverse problem, we generalize the algorithm given in [4] by weighting the individual error terms. We establish the existence of a global optimum, characterize its properties, and present the conditions under which the optimum is unique. Furthermore, we develop an efficient numerical algorithm for approximating the global optimum. We term our localization approach UNLOC, unfolding-based localization, since the same type of optimization has been previously adopted in the literature of multidimensional scaling as "unfolding" [10]. We validate the effectiveness of UNLOC using both synthetic and real experimental data, showing that appropriate incorporation of weights can produce much improved localization than unweighted UNLOC.

The rest of this paper is organized as follows. In Section 2, we present the forward model and formulate the localization problem as an inverse problem. In Section 3, we develop UNLOC using the optimization of a nonlinear stress function, analyze the existence and uniqueness of global optimum, and present an efficient algorithm to search for such optimum. Numerical simulations are shown in Section 4, and experiments with Android hardware are shown in Section 5. A brief discussion and concluding remarks are given in Section 6.

2. **Target Localization as an Inverse Problem**

We start by formulating target localization as an inverse problem. Given a set of m anchors whose locations are known, and (noisy) distance measurements from each of the anchors to a target at unknown location. The problem is to estimate, or in other words to infer, the location of the target.

2.1. **Forward Model.** In a typical localization problem, several sensors at known locations (anchors) are used to estimate the location of a target node at an unknown location. Localization can be range-based, direction-based, or array-based. In range-based models, distances between the target node and anchors can be estimated using time delays or power loss. In each case, each node device can be a transmitter or a receiver. For simplicity, we consider range-based localization with distances being estimated using power-loss estimates between the transmitting target node and receiving anchors [26]. This is modeled as the following forward problem.

In a p-dimensional space, let $\boldsymbol{y}_i \in \mathbb{R}^p$ denote the (known) location of the ith anchor ($i = 1, 2, \ldots, m$) and δ_i represent the noisy distance measured between the ith anchor and the target ($i = 1, 2, \ldots, m$). The *forward* model of the problem describes how the measured distance depends on actual anchor-to-target distance and noise, and can generally be expressed as

$$\delta_i = f(d_i, \xi_i), \tag{1}$$

for $i = 1, 2, \ldots, m$, where $d_i = d(\boldsymbol{x}, \boldsymbol{y}_i) = \|\boldsymbol{x} - \boldsymbol{y}_i\|_2 = \sqrt{(\boldsymbol{x} - \boldsymbol{y}_i)^\top (\boldsymbol{x} - \boldsymbol{y}_i)}$ denotes the (Euclidean) distance between the target node and the i-th anchor, ξ_i represents noise, and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function that models the dependence of (noisy) output on input signals. Without loss of generality, we assume that the function $f(d, \xi)$ has unit derivative in the absence of noise, that is, $\frac{\partial f}{\partial d}|_{\xi=0} = 1$. Note that f can generally be made to satisfy this property by scaling $f \to cf$ under an appropriately chosen constant c.

2.2. **Common Forms of the Forward Model.** In sensor network applications, the forward model generally takes the following form

$$\delta_i = f(d_i, \xi_i) = (d_i^\gamma + \xi_i)^{1/\gamma}, \tag{2}$$

where the parameter $\gamma > 0$ reflects the type of observed signals: $\gamma = 1$ corresponds to direct measure of distance, and $\gamma \neq 1$ usually corresponds to indirect measure of distance. For example, when transmission power is used as a proxy for distance, the value of $\gamma = 2$ in free space as derived from the path-loss formula. As we show later in the paper, the complexity and difficulty of the localization problem as well as the quality of the solution are heavily influenced by the particular forward model.

2.3. **The Inverse Problem of Localization.** As discussed above, measurement of the anchor-to-target distances is prone to noise, and generally difficult to model precisely [12]. Using noisy distance data, the estimation of the location of the target node can be formulated as an inverse problem as follows.

Given the location of m anchors denoted by $\{\boldsymbol{y}_i \in \mathbb{R}^p\}_{i=1,\ldots,m}$ in p-dimensional space, and noisy distance measures $\{\delta_i \in \mathbb{R}\}_{i=1,\ldots,m}$, the inverse problem is to estimate the location of the target, $\boldsymbol{x} \in \mathbb{R}^p$.

Note that even given the form of forward model as in (2), and under iid noise, the quality of localization can vary drastically depending on the parameter γ. This is because in order for the inverse problem to be solvable, one typically defines an objective function as the sum of squared difference between $\delta_i^2 - d_i^2$ (see next section for details). When $\gamma = 2$, the objective function effectively measures the sum of squared noise, and is expected to provide nearly optimal localization; on the other hand, under other values of γ or when the noise are not iid, then such conclusion will likely not hold. For example, for the case of $\gamma = 1$ and iid noise, the difference $\delta_i^2 - d_i^2 = \xi_i(2d_i + \xi_i) \approx 2d_i\xi_i$ when the noise is much smaller compared to the actual distance. Thus, even though the noise in the forward model are iid, when solving an inverse problem the effective "noise" can in fact be range-dependent and present a major challenge.

3. UNLOC: Unfolding-based Localization as an Optimization Problem

To solve the target localization problem, we consider the optimization of a weighted objective function

$$J(\boldsymbol{x}, Y; \boldsymbol{\delta}; \boldsymbol{w}) = \sum_{i=1}^{m} w_i \left(d(\boldsymbol{x}, \boldsymbol{y}_i)^2 - \delta_i^2 \right)^2. \tag{3}$$

This type of function commonly occurs in the field of multidimensional scaling (MDS) and is known as a (weighted) stress function [7]. The value of the stress function depends on $\boldsymbol{x}$ (unknown target location), $Y = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m]^\top \in \mathbb{R}^{m \times p}$ (known anchor locations), $\boldsymbol{\delta}$ (measured anchor-to-target distances), and a set of weights defined by the vector $\boldsymbol{w} = [w_1, \ldots, w_m]^\top$. For a given localization problem (fixed Y and $\boldsymbol{\delta}$), the stress function quantifies the discrepancy between the observed and recovered anchor-to-target distances, weighted by the coefficients $\{w_i\}$. Thus, minimization of the stress function over $\boldsymbol{x}$ produces an estimated target location. Throughout the paper, we assume that the weights are all positive, that is $w_i > 0$ $(\forall i)$, or simply written as $\boldsymbol{w} > \boldsymbol{0}$.

Stated briefly, the proposed unfolding-based localization (UNLOC) is based on solving the following stress optimization problem:

$$\min_{\boldsymbol{x} \in \mathbb{R}^p} J(\boldsymbol{x}, Y; \boldsymbol{\delta}; \boldsymbol{w}), \tag{4}$$

where the minimizer $\boldsymbol{x}^*$, if any, produces an estimate of the target location.

3.1. Conversion into a constrained quadratic optimization problem.

To solve the weighted stress optimization problem in (4), we follow the approach in [4] to convert it into a constrained quadratic optimization that, despite being nonlinear and nonconvex, yields a global minimizer that can be explicitly characterized and constructed using efficient numerical algorithms.

We start by defining an extended vector $\boldsymbol{z} = [\boldsymbol{x}^\top, \alpha]^\top \in \mathbb{R}^{p+1}$. The weighted stress optimization problem in (4) can be equivalently expressed as the following constrained quadratic optimization problem:

$$\min_{\boldsymbol{z} \in \mathbb{R}^{p+1}} \{q(\boldsymbol{z}) : c(\boldsymbol{z}) = 0\}, \tag{5}$$

where

$$\begin{cases} q(\boldsymbol{z}) = \|W(M\boldsymbol{z} - \boldsymbol{b})\|^2 = (WM\boldsymbol{z} - W\boldsymbol{b})^\top (WM\boldsymbol{z} - W\boldsymbol{b}), \\ c(\boldsymbol{z}) = \boldsymbol{z}^\top D \boldsymbol{z} + 2\boldsymbol{f}^\top \boldsymbol{z}, \end{cases} \tag{6}$$

and $W = \begin{bmatrix} \sqrt{w_1} & & \\ & \ddots & \\ & & \sqrt{w_m} \end{bmatrix}$, $M = \begin{bmatrix} -2\boldsymbol{y}_1^\top & 1 \\ \vdots & 1 \\ -2\boldsymbol{y}_m^\top & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} \delta_1^2 - \|\boldsymbol{y}_1\|^2 \\ \vdots \\ \delta_m^2 - \|\boldsymbol{y}_m\|^2 \end{bmatrix}$, $D = \begin{bmatrix} I_{p\times p} & \boldsymbol{0}_{p\times 1} \\ \boldsymbol{0}_{1\times p} & 0 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} \delta_1^2 - \|\boldsymbol{y}_1\|^2 \\ \vdots \\ \delta_m^2 - \|\boldsymbol{y}_m\|^2 \end{bmatrix}$, $\boldsymbol{f} = \begin{bmatrix} \boldsymbol{0}_{p\times 1} \\ -0.5 \end{bmatrix}$.

The quadratic optimization problem in (5), despite being nonconvex, indeed admits a global minimizer that can be explicitly characterized and constructed via efficient algorithms under quite mild assumptions [15], which we discuss below, for our specialized localization problem, in detail.

3.2. **Existence of the global minimizer.** Here we show that the quadratic optimization problem in (5) always has a global minimizer, by applying results from Ref. [15] [Theorem 2.1 therein], which yields the lemma below.

Lemma 1. *Suppose that* $\boldsymbol{w} > \mathbf{0}$*. Then the quadratic optimization problem* (5) *has a global minimizer.*

Proof. Let $A = \nabla^2 q(\boldsymbol{z}) = 2M^\top W^\top WM$ and $C = \nabla^2 c(\boldsymbol{z}) = 2D$. Applying Theorem 2.1 of Ref. [15], the optimization problem (5) has a global minimizer if the feasible set $\{\boldsymbol{z} \in \mathbb{R}^{p+1} : c(\boldsymbol{z}) = 0\}$ is nonempty, and that $\boldsymbol{z} \neq \mathbf{0}$ and $\boldsymbol{z}^\top C\boldsymbol{z} = 0$ implies that $\boldsymbol{z}^\top A\boldsymbol{z} > 0$.

Suppose that $\boldsymbol{z} = [\boldsymbol{x}^\top, \alpha]^\top$ where $\alpha \in \mathbb{R}$, we first show that the feasible set is always nonempty. Since $c(\boldsymbol{z}) = \|x\|^2 - \alpha$, the vector $\boldsymbol{z} = \mathbf{0}$ is always an element of the feasible set. Next, suppose that $\boldsymbol{z} = [\boldsymbol{x}^\top, \alpha]^\top \neq \mathbf{0}$ where $\alpha \in \mathbb{R}$, and that $\boldsymbol{z}^\top C\boldsymbol{z} = 2\|\boldsymbol{x}\|^2 = 0$. Thus $\boldsymbol{x} = \mathbf{0}$ and $\alpha \neq 0$, and the vector $WM\boldsymbol{z} = WM[\boldsymbol{x}^\top, \alpha]^\top = WM[\mathbf{0}^\top, \alpha]^\top = \alpha[\sqrt{w_1}, \ldots, \sqrt{w_m}]^\top$. Therefore $\boldsymbol{z}^\top A\boldsymbol{z} = 2\|WM\boldsymbol{z}\|^2 = 2\alpha^2 \sum_{i=1}^m w_i > 0$. □

3.3. **Characterization of the global minimizer.** Having shown the existence of a global minimizer, we now characterize it by providing explicit conditions that must be satisfied by a global minimizer. Such characterization is important for algorithmic construction of the (approximate) minimizer.

Lemma 2. *A vector* $\boldsymbol{z}^* \in \mathbb{R}^{p+1}$ *is a global minimizer of problem* (5) *if and only if there exists a multiplier* $\lambda^* \in \mathbb{R}$ *such that the following optimality conditions hold:*

$$\begin{cases} (a)\ \boldsymbol{z}^{*\top} D\boldsymbol{z}^* + 2\boldsymbol{f}^\top \boldsymbol{z}^* = 0, \\ (b)\ (M^\top W^\top WM + \lambda^* D)\boldsymbol{z}^* = M^\top W^\top W\boldsymbol{b} - \lambda^* \boldsymbol{f}, \\ (c)\ M^\top W^\top WM + \lambda^* D \succeq 0, \end{cases} \tag{7}$$

where the inequality in (c) is interpreted as the left-hand-side matrix being positive semidefinite.

Proof. It is straightforward to verify that the Hessian matrix $\nabla^2 c(\boldsymbol{z}) = 2D \neq 0$. The condition

$$\min_{\boldsymbol{z}\in\mathbb{R}^{p+1}} \{c(\boldsymbol{z})\} < 0 < \max_{\boldsymbol{z}\in\mathbb{R}^{p+1}} \{c(\boldsymbol{z})\} \tag{8}$$

is always satisfied since $c(\boldsymbol{z}) = \boldsymbol{z}^\top D\boldsymbol{z} + 2\boldsymbol{f}^\top \boldsymbol{z} = \|\boldsymbol{x}\|^2 - \alpha$ can be made arbitrarily large (or small) by controlling the value of α.

The conclusion of the lemma then follows by applying Theorem 3.2 of Ref. [15], noting that in our optimization problem (5), $\nabla q(\boldsymbol{z}) = 2M^\top W^\top W(M\boldsymbol{z} - \boldsymbol{b})$, $\nabla^2 q(\boldsymbol{z}) = 2M^\top W^\top WM$, $\nabla c(\boldsymbol{z}) = 2(D\boldsymbol{z} + \boldsymbol{f})$, and $\nabla^2 c(\boldsymbol{z}) = 2D$. □

Note that the conditions (7) in the above lemma, which are necessary conditions for the global minimizer to satisfy, are often referred to as the Karush-Kuhn-Tucker (KKT) conditions of the problem [5, 17].

3.4. **Uniqueness of the global minimizer.** The following lemma is a direct corollary of Theorem 4.1 of [15] regarding the uniqueness of the global minimizer of problem (5).

Lemma 3. *Consider the optimization problem* (5)*. Suppose that* $\boldsymbol{z}^*$ *satisfies the optimality conditions in* (7) *for some* $\lambda^* \in \mathbb{R}$*. Then* $\boldsymbol{z}^*$ *is the unique global minimizer of problem* (5) *if the inequality of* (7)*-(c) is strict: that is, if the matrix* $M^\top W^\top WM + \lambda^* D$ *is positive definite.*

For this work, we assume that the condition in Lemma 3 is satisfied, that is, the matrix $M^\top W^\top WM + \lambda^* D$ is positive definite, which guarantees uniqueness of the solution. In practice, due to noise other imperfections, we expect such condition to always hold.

3.5. **Explicit computation of the global minimizer.** Finally, we adopt the general computation proposed in Ref. [15] to develop a specialized algorithm for iteratively approximating the global minimizer of problem (5).

Consider the set

$$I_{PD} = \{\lambda \in \mathbb{R} : M^\top W^\top WM + \lambda D \succ 0\}, \tag{9}$$

which we prove to be nonempty for our optimization problem.

Lemma 4. *Consider the optimization problem* (5)*. The set* I_{PD} *as defined in* (9) *is a nonempty interval of the form* (a, ∞)*, where* $a \geq -\min \Lambda(M^\top W^\top WM)$ *with* $\Lambda(\cdot)$ *denoting the set of eigenvalues of the corresponding matrix.*

Proof. First, by Theorem 2.2 of Ref. [15], the matrix $M^\top W^\top WM + \lambda D$ is positive definite for some $\lambda \in \mathbb{R}$ if and only if the statement below is true:

$$\left.\begin{array}{c} \boldsymbol{z} \neq \boldsymbol{0} \\ \boldsymbol{z}^\top C \boldsymbol{z} = 0 \end{array}\right\} \Rightarrow \boldsymbol{z} A \boldsymbol{z} > 0, \tag{10}$$

where $A = \nabla q(\boldsymbol{z}) = 2M^\top W^\top WM$ and $C = \nabla c(\boldsymbol{z}) = 2D$. Such statement was shown to be true in the proof of Lemma 1.

Next, since A and C are both symmetric, Theorem 5.1 of Ref. [15] implies that I_{PD} is an interval. Since matrices $M^\top W^\top WM$ and λD are both symmetric, and the eigenvalues of D are 1 (multiplicity p) and 0 (simple eigenvalue), Weyl's eigenvalue inequality implies that: (1) if $M^\top W^\top WM + \lambda_1 D \succ 0$ and $\lambda_2 > \lambda_1$, then $M^\top W^\top WM + \lambda_2 D \succ 0$, and (2) the choice of $\lambda = -\min \Lambda(M^\top W^\top WM)$ renders the smallest eigenvalue of $M^\top W^\top WM + \lambda D$ nonpositive. □

In order to find a vector $\boldsymbol{z}$ that satisfies the global optimality conditions in (7), we define

$$\begin{cases} \boldsymbol{z}(\lambda) = (M^\top W^\top WM + \lambda D)^{-1}(M^\top W^\top W\boldsymbol{b} - \lambda \boldsymbol{f}), \\ \phi(\lambda) = \boldsymbol{z}(\lambda)^\top D\boldsymbol{z} + 2\boldsymbol{f}^\top \boldsymbol{z}(\lambda). \end{cases} \tag{11}$$

Following Theorem 5.1 of Ref. [15], the scalar function $\phi(\lambda)$ is strictly decreasing on the nonempty interval I_{PD}. The task of finding $\lambda^* \in I_{PD}$ such that $\phi(\lambda^*) = 0$ is achieved by searching over the interval $(-\min \Lambda(M^\top W^\top WM), \infty)$, for example, using bisection starting with a sufficiently large value [15]. Once such an λ^* is found (or approximated to within some prescribed tolerance), we compute z^* by (11), as $z^* = z(\lambda^*)$.

4. Numerical Simulations

To benchmark the effectiveness of the proposed UNLOC algorithms for localization, we conduct tests and computations on both synthetic data (this section) and experimental data (next section).

In this section, we focus on synthetic data generated by numerical simulations. We fix the number of anchors, $m = 5$ (the same number is used in our lab experiments as discussed in the next section). In all numerical examples, the anchors and the target node are placed randomly in the unit square $[0, 1]^2$, and the noisy distance data $\{\delta_i\}$ are obtained by simulating the forward model (2). Different parameter choices and noise distributions are considered, and the performance of UNLOC is evaluated in each case.

4.1. Example 1.

In the first example, we consider $\gamma = 2$ in Eq. (2) with each ξ_i independently drawn from the normal distribution $\mathcal{N}(0, \sigma^2)$, to give:

$$\delta_i^2 = d(\boldsymbol{x}, \boldsymbol{y}_i)^2 + \xi_i, \tag{12}$$

which implies that the weighted stress function in (3) now becomes

$$J(\boldsymbol{x}, Y; \boldsymbol{\delta}; \boldsymbol{w}) = \sum_{i=1}^{m} w_i \xi_i^2, \tag{13}$$

and the minimization of J attempts to minimize the weighted squared noise (as determined by the forward model).

4.2. Example 2.

In the second example, we consider $\gamma = 1$ in Eq. (2) with each ξ_i independently drawn from the normal distribution $\mathcal{N}(0, \sigma^2)$, thus

$$\delta_i^2 = (d(\boldsymbol{x}, \boldsymbol{y}_i) + \xi_i)^2, \tag{14}$$

which implies that the weighted stress function in (3) now becomes

$$J(\boldsymbol{x}, Y; \boldsymbol{\delta}; \boldsymbol{w}) = \sum_{i=1}^{m} w_i (2d(\boldsymbol{x}, \boldsymbol{y}_i)\xi_i + \xi_i^2)^2, \tag{15}$$

and so the minimization of J attempts to minimize the weighted squared noise (as determined by the forward model).

4.3. **Example 3.** Finally, in example 3 we focus on the same noise model as in Example 2, with the noise drawn from the Laplace distribution instead of the normal distribution. The pdf of the Laplace distribution is given by

$$p(\xi) = \frac{1}{\sqrt{2}\sigma} \exp(-\sqrt{2}\sigma|\xi|), \tag{16}$$

where σ^2 is the variance of the distribution. The Laplace distribution is more "heavy-tailed" than the normal distribution with the same variance. The motivation to test such a distribution is that this form of noise distribution commonly occurs in several applications [13], including population dynamics [20], growth of companies [18], mechanical vibrations [1], and turbulent flows [24].

Note that in both Example 2 and Example 3, even though the noise ξ_i that enters the model might be identical and independent from each other, in terms of the weighted optimization the effective "noise" is in fact range-dependent, being the difference between δ_i^2 and $d(\boldsymbol{x}, \boldsymbol{y}_i)^2$. Such scenario of range-dependent noise commonly observed in RSSI-based localization [25, 26].

We first investigate how the accuracy of localization depends on the choice of weights. In particular, we consider weights of the form

$$w_i = \delta_i^{\beta}, \tag{17}$$

with a single parameter β. When $\beta = 0$, all the weights w_i are the same and independent of δ_i; on the other hand, when $\beta > 0$ ($\beta < 0$), more weight is given to larger (smaller) measured anchor-to-target distances. As shown in Fig. 2, the choice of weights have nonnegligible impact on localization. In particular, the "localization error", computed as the mean square distance between the true and estimated target location obtained from a large number (10, 000) of independent simulations, is minimized at $\beta \approx 0$ (unweighted UNLOC) only in Example 1; in both Example 2 and Example 3, the localization errors using unweighted UNLOC is generally larger than those using weighted UNLOC with negative β, with the optimal value of $\beta \approx -2$.

To further explore the difference in localization outcomes, in Fig. 3, we compare the performance of unweighted UNLOC ($\beta = 0$) and "optimally" weighted UNLOC (with β selected to minimize the $\sigma = 0.1$ curve in each example). It is clear that weighted UNLOC produces better localization than their unweighted counterpart, and such improvement is more significant when the noise is range-dependent (Examples 2 and 3).

5. Experimental Validation

In addition to the validation of the proposed algorithm on synthetic data, we have also conducted lab experiments to test our algorithm on real-life data. In each experiment, we used Android devices as transceivers. In particular, $m = 5$ Android devices were designated as anchors and placed in fixed positions. One Android device was designated as the target and placed at a fixed location (unknown to the algorithm). Bluetooth links were established between the target

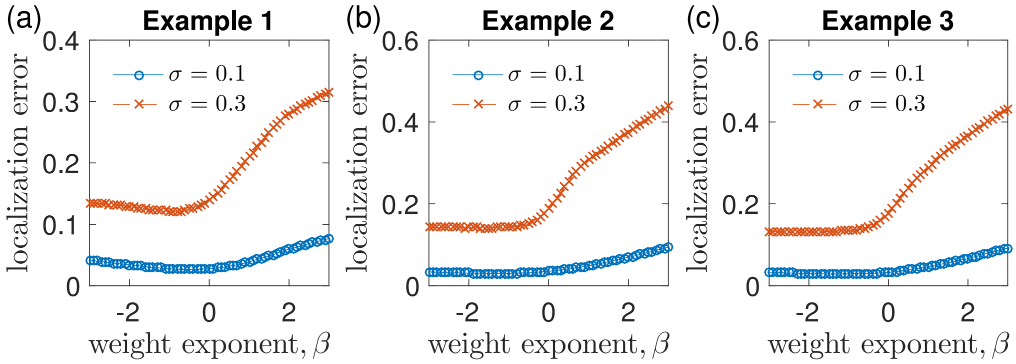

Figure 2. Localization error vs weights under two noise stds: $\sigma = 0.1$ and $\sigma = 0.3$. The distribution of noise in Examples 1 and 2 are normal, whereas in Example 3 it is Laplace, as in Eq. (16). The results are obtained from 10^4 independent simulations.

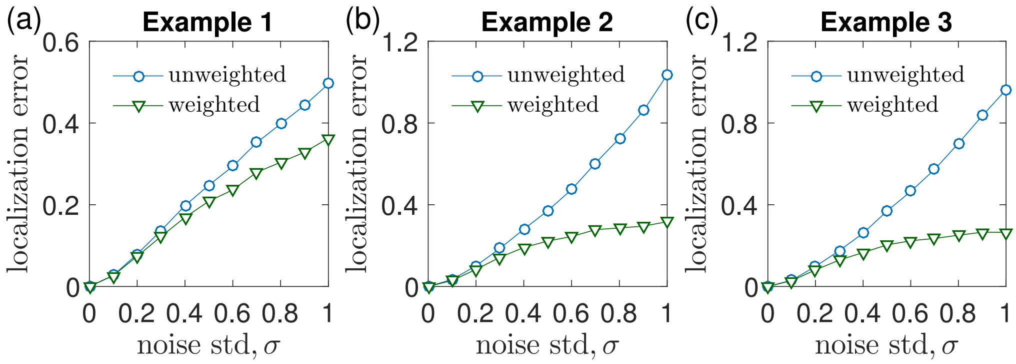

Figure 3. Localization error vs noise std. For the weighted UNLOC, the value of β in each example is the one that minimizes the $\sigma = 0.1$ localization error curve in Fig. 2, giving: $\beta = -0.5$ (Example 1), $\beta = -1.3$ (Example 2), and $\beta = -2.2$ (Example 3), respectively. Each data point represents an averaged over $10,000$ independent simulations.

device and each anchor and received signal strength indication (RSSI) was used to estimate the distance between the target and each anchor [2].

We show the result of localization in Fig. 4 together with the sensor field for all four experiments. In particular, we found that weighted UNLOC provides better localization than unweighted UNLOC, where the weights are determined using Eq. (17) as $w_i = \delta^\beta$ with the exponent selected according to minimal localization error in each experiment, giving $\beta = -2.4$ (Exp. 1), $\beta = 3.0$ (Exp. 2), $\beta = -2.8$ (Exp. 3), and $\beta = -1.7$ (Exp. 4), respectively.

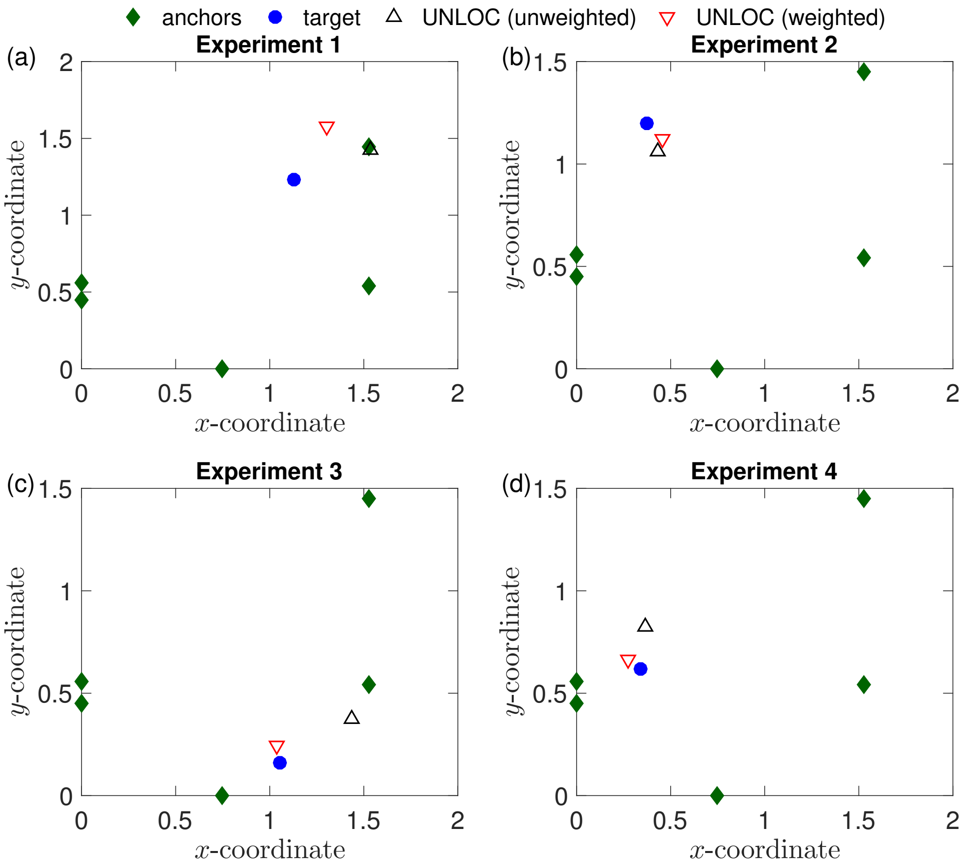

Figure 4. Localization results from experimental data. In all experiments, the estimated location is reasonably close to the actual target location, and weighted UNLOC provides better estimate of the target location than unweighted UNLOC.

6. Discussion and Conclusions

In this paper, we consider the problem of target localization, where noisy distance measurements between the anchors at known locations and a target node at an unknown location are used to find the coordinates of the target node. We formulate the problem as an inverse problem, where the forward model describes how the noisy distance output depends on the location of anchors (known), the target (unknown), and noise (unknown). To solve the inverse problem, that is, to estimate the location of the target from noisy distance data, we consider the optimization of a weighted stress function, and study the conditions under which a global optimum exists and is unique, and further develop efficient algorithms for finding such optimum. We refer to our data-enabled computational approach for target location as UNLOC, combining the words "unfolding" and "localization", the former being a widely used term in the field of multidimensional scaling for the proposed optimization.

The UNLOC algorithm solves a non-convex optimization problem, where a stress function is minimized, subject to constraints. We showed that under mild conditions, a unique minimizer exists. Further, by incorporating weights into the problem, we can solve the optimization problem for unknown noise models. We perform numerical simulations to demonstrate the applications of UNLOC, using both synthetic data and data collected in a localization experiment with Android devices. The results show that UNLOC can accurately localize a target even with relatively large noise, and the accuracy can be further improved by adjusting the weights in the optimization function.

Future work includes extending UNLOC to the case of multiple target nodes, incorporating anchor-to-anchor distances, consideration of uncertainties in anchor locations, and *ad-hoc* scenarios where there are no anchors at all. We expect our recent work of BiFold to be applicable to some of these problems, being able to simultaneously accounting for multiple types of distance measures [11]. In addition, in some applications there are often constraints or prior information on the possible location of the target. We foresee such constraints or prior information be accounted for by extending the optimization problem to incorporate additional (regularization) terms [3,21–23]. Furthermore, given that the forward model is often unknown, it is of importance to be able to optimize the weights. Our ongoing work includes the development of a cross-validation framework for the selection of weights, which can also provide valuable information about the uncertainty of the localization solution.

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