ALL CONTINUOUS FUNCTIONS ON $[a, b]$ ARE RIEMANN-INTEGRABLE

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1. The statement of the theorem

Theorem. *All real-valued continuous functions on the closed and bounded interval $[a, b]$ are Riemann-integrable.*

1.1. We need to know the following theorems.

(i) Monotone convergence theorem
(ii) Extreme value theorem
(iii) Heine-Cantor theorem

1.2. We need to know the following concepts.

(i) The lower sum $\mathcal{L}_{\mathcal{P}}$, upper sum $\mathcal{U}_{\mathcal{P}}$ and Riemann sum $\mathcal{R}_{\mathcal{P}}$ for a partition $\mathcal{P}$ of $[a, b]$.
(ii) A sequence of dyadic partitions $\{\mathcal{P}_{2^n}\}_{n=0}^{\infty}$ of $[a, b]$.
(iii) Corresponding sums $\mathcal{L}_{2^n}, \mathcal{U}_{2^n}$ and $\mathcal{R}_{2^n}$ for a dyadic partition $\mathcal{P}_{2^n}$.
(iv) The refinement $\mathcal{P}'$ of $\mathcal{P}$.
(v) Uniform continuity.

2. Modus Operandi

In this manuscript we follow the following five steps:

(I) We show that $\lim_{n\to\infty} \mathcal{L}_{2^n}$ and $\lim_{n\to\infty} \mathcal{U}_{2^n}$ exist. We will use the monotone convergence theorem for this. In particular, we show that $\{\mathcal{L}_{2^n}\}_{n=0}^{\infty}$ is bounded and monotone increasing i.e. $\mathcal{L}_{2^n} \leq \mathcal{L}_{2^{n+1}}$.
(II) We will show that $\lim_{n\to\infty}(\mathcal{U}_{2^n} - \mathcal{L}_{2^n}) = 0$.
To this effect we use the fact that f is uniformly continuous on $[a, b]$, which follows from Heine-Cantor theorem.
We further conclude from (I) that $\lim_{n\to\infty} \mathcal{U}_{2^n} = \lim_{n\to\infty} \mathcal{L}_{2^n}$.
We denote this limit by I, i.e. $\lim_{n\to\infty} \mathcal{L}_{2^n} = I$.
(III) Now, for a general partition $\mathcal{P}$ of $[a, b]$, we prove that $\mathcal{L}_{\mathcal{P}} \leq \mathcal{L}_{\mathcal{P}'} \leq \mathcal{U}_{\mathcal{P}'} \leq \mathcal{U}_{\mathcal{P}}$, where $\mathcal{P}'$ is a refinement of $\mathcal{P}$. The proof of the fact that $\mathcal{L}_{\mathcal{P}} \leq \mathcal{L}_{\mathcal{P}'}$ is very similar to the proof of $\mathcal{L}_{2^n} \leq \mathcal{L}_{2^{n+1}}$.
(IV) We show $\lim_{\|P\|\to 0}(\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}'}) = 0$. This is done by realizing that $|\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}'}| \leq \mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}}$, but we show $\lim_{n\to\infty} \mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}} = 0$.
(V) Finally we show $\lim_{\|\mathcal{P}\|\to 0} \mathcal{R}_{\mathcal{P}} = I$.
This follows from the observation: $|\mathcal{R}_{\mathcal{P}} - I| \leq |\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}'}| + |\mathcal{R}_{\mathcal{P}'} - \mathcal{L}_{2^n}| + |\mathcal{L}_{2^n} - I|$.

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3. Preliminaries

Theorem (Monotone convergence theorem)**.** *Every bounded and monotone sequence is convergent.*

Theorem (Extreme value theorem)**.** *A continuous function f on a closed and bounded (nonempty) interval $[a, b]$ attains its extreme values.*

Definition (Continuity at a point)**.** *We say that the function f is continuous at $x_0 \in [a, b]$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that*
whenever $|x - x_0| < \delta$
thenever[1] $|f(x) - f(x_0)| < \epsilon$.

(*Note, that this δ may not work for a different x_0.*)

Definition (Continuity on an interval)**.** *We say that a function f is continuous on $[a, b]$ if it is continuous at each point in $[a, b]$.*

Definition (Uniform continuity)**.** *We say that that the function f is uniformly continuous on $[a, b]$ if for each $\epsilon > 0$ there exists a $\delta > 0$ such that*

whenever $|x_1 - x_2| < \delta$

thenever $|f(x_1) - f(x_2)| < \epsilon$.

for all $x_1, x_2 \in [a, b]$.

(*Note, that the **same** δ works **for all** $x_1, x_2 \in [a, b]$. The uniform continuity is much stronger condition than the continuity. All uniformly continuous function are continuous, but all continuous functions are not uniformly continuous.*)

Theorem (Heine-Cantor)**.** *Every continuous function on **a closed and bounded** interval is uniformly continuous.*

Definition (Partition)**.** *We define a partition $\mathcal{P}$ of $[a, b]$ as a finite sequence of numbers $\{x_0, x_1, x_2, \ldots, x_n\}$ such that $x_0 = a < x_0 < x_1 < x_2 \cdots < x_n = b$.*

Definition (Refinement of a partition)**.** *We say that $\mathcal{P}'$ is a refinement of $\mathcal{P}$ if $\mathcal{P}'$ contains all the points of $\mathcal{P}$, with few more points.*

Definition (Norm of a partition)**.** *Norm of a partition $\mathcal{P}$ denoted by $\|\mathcal{P}\|$ is defined as $\|\mathcal{P}\| = \max_{1 \le i \le n} |x_i - x_{i-1}|$.*

Definition (Sequence of dyadic partition of an interval)**.** *We define the partition $\mathcal{P}_{2^0} = \{a, b\}$. We obtain the refinement $\mathcal{P}_{2^{n+1}}$ of $\mathcal{P}_{2^n}$ by adding midpoints of subintervals from the previous partition $\mathcal{P}_{2^n}$.*

$$
\begin{aligned}
\textit{i.e.} \quad & \mathcal{P}_{2^0} = \{a, b\} \\
& \mathcal{P}_{2^1} = \{a, \frac{a+b}{2}, b\} \\
& \mathcal{P}_{2^2} = \{a, a + \frac{b-a}{4}, a + 2\frac{b-a}{4}, a + 3\frac{b-a}{4}, b\} \\
& \ldots
\end{aligned}
$$

[1] the word "thenever" is copyrighted by Prashant Athavalé

Thus, we construct a sequence of partitions $\{\mathcal{P}_{2^n}\}_{n=0}$.
Note, that there are 2^n subintervals in the partition $\mathcal{P}_{2^n}$ of $[a, b]$, each of length $\left(\frac{b-a}{2^n}\right)$.

Definition (Riemann sum $\mathcal{R}_{\mathcal{P}}$). *We define the Riemann sum $\mathcal{R}_{\mathcal{P}}$ as*

$$\mathcal{R}_{\mathcal{P}} := \sum_{i=1}^{n} f(x_i^*) \triangle x_i,$$

where x_i^* is ***any*** *point in the i^{th} subinterval $[x_{i-1}, x_i]$ of the partition $\mathcal{P}$ of $[a, b]$.*

Definition (Lower sum). *We define the lower sum $\mathcal{L}_{\mathcal{P}}$ as*

$$\mathcal{L}_{\mathcal{P}} := \sum_{i=1}^{n} f_i^{min} \triangle x_i,$$

where f_i^{min} is the minimum value of the function f on the i^{th} subinterval $[x_{i-1}, x_i]$ of the partition $\mathcal{P}$ of $[a, b]$.

Note, we know from the extreme value theorem that there exists a point $x_i^{min} \in [x_{i-1}, x_i]$ such that $f(x_i^{min}) = f_i^{min}$.

Definition (Upper sum). *We define the upper sum $\mathcal{U}_{\mathcal{P}}$ as*

$$\mathcal{U}_{\mathcal{P}} := \sum_{i=1}^{n} f_i^{max} \triangle x_i,$$

where f_i^{max} is the maximum value of the function f on the i^{th} subinterval $[x_{i-1}, x_i]$ of the partition $\mathcal{P}$ of $[a, b]$.

Again, we know from the extreme value theorem that there exists a point $x_i^{max} \in [x_{i-1}, x_i]$ such that $f(x_i^{max}) = f_i^{max}$.

Remark. *Special cases of the lower, upper, and Riemann sums for the dyadic partition $\mathcal{P}_{2^n}$ are denoted by $\mathcal{L}_{2^n}$,$\mathcal{U}_{2^n}$ and $\mathcal{R}_{2^n}$. An important fact is that for dyadic partition $\triangle x_i = \left(\frac{b-a}{2^n}\right)$ for all subintervals of $[a, b]$. Thus,*

$$\mathcal{L}_{2^n} := \left(\frac{b-a}{2^n}\right) \sum_{i=0}^{2^n} f_i^{min}, \quad \mathcal{U}_{2^n} := \left(\frac{b-a}{2^n}\right) \sum_{i=0}^{2^n} f_i^{max}, \quad \mathcal{R}_{2^n} := \left(\frac{b-a}{2^n}\right) \sum_{i=0}^{2^n} f(x_i^*).$$

Definition. *We define the Riemann integral as the limit of the Riemann sum i.e.*

$$\int_a^b f(x)\,dx = \lim_{\|\mathcal{P}\| \to 0} \mathcal{R}_{\mathcal{P}} = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=0}^{n} f(x_i^*) \triangle x_i,$$

where x_i^* is a point in the i^{th} subinterval of $[a, b]$, i.e. $x_i^* \in [x_{i-1}, x_i]$.

4. Results about the sequence of dyadic partition $\{\mathcal{P}_{2^n}\}_{n=0}^{\infty}$

Lemma 1. *If f is a continuous function on $[a, b]$ then the sequences $\{\mathcal{L}_{2^n}\}_{n=1}^{\infty}$ and $\{\mathcal{U}_{2^n}\}_{n=1}^{\infty}$ are bounded.*

Proof. Let $M := \max_{x\in[a,b]} f(x)$ and $m := \min_{x\in[a,b]} f(x)$. By the extreme value theorem m and M exist and are finite real numbers. The following is true for all i and n:

$$m \le f_i^{min} \le f_i^{max} \le M$$
$$\sum_{i=1}^{2^n} m \le \sum_{i=1}^{2^n} f_i^{min} \le \sum_{i=1}^{2^n} f_i^{max} \le \sum_{i=1}^{2^n} M$$
$$\left(\frac{b-a}{2^n}\right)\sum_{i=1}^{2^n} m \le \left(\frac{b-a}{2^n}\right)\sum_{i=1}^{2^n} f_i^{min} \le \left(\frac{b-a}{2^n}\right)\sum_{i=1}^{2^n} f_i^{max} \le \left(\frac{b-a}{2^n}\right)\sum_{i=1}^{2^n} M$$
$$m(b-a) \le \mathcal{L}_{2^n} \le \mathcal{U}_{2^n} \le M(b-a)$$

□

Now if we could show that the sequence $\{\mathcal{L}_{2^n}\}_{n=0}^{\infty}$ is monotone increasing, then we can conclude from the monotone convergence theorem that sequence $\{\mathcal{L}_{2^n}\}_{n=0}^{\infty}$ must converge. To show that $\{\mathcal{L}_{2^n}\}_{n=0}^{\infty}$ is monotone increasing we need to show that $\mathcal{L}_{2^{n+1}} \ge \mathcal{L}_{2^n}$.

Lemma 2. *For the dyadic sequence of partitions of* $[a,b]$, *the sequence* $\{\mathcal{L}_{2^n}\}_{n=1}^{\infty}$ *is monotone increasing.*

Proof. Let us divide the i^{th} subinterval $[x_{i-1}, x_i]$ in half to get two subintervals, the left subinterval: $[x_{i-1}, \frac{x_{i-1}+x_i}{2}]$ and the right subinterval: $[\frac{x_{i-1}+x_i}{2}, x_i]$. Let $f_{i,left}^{min}$ be the minimum value of f on the left subinterval and $f_{i,right}^{min}$ be the minimum value of f on the right subinterval. As f_i^{min} is the minimum over the bigger subinterval $[x_{i-1}, x_i]$, we must have

$$f_{i,left}^{min} \ge f_i^{min} \quad \text{as well as} \quad f_{i,right}^{min} \ge f_i^{min}.$$

Thus, adding these equations and dividing by 2 we get

$$\frac{1}{2}(f_{i,left}^{min} + f_{i,right}^{min}) \ge f_i^{min}$$
$$\sum_{i=1}^{2^n} \frac{1}{2}(f_{i,left}^{min} + f_{i,right}^{min}) \ge \sum_{i=1}^{2^n} f_i^{min}$$
$$\left(\frac{b-a}{2^n}\right)\sum_{i=1}^{2^n} \frac{1}{2}(f_{i,left}^{min} + f_{i,right}^{min}) \ge \left(\frac{b-a}{2^n}\right)\sum_{i=1}^{2^n} f_i^{min}$$
$$\left(\frac{b-a}{2^{n+1}}\right)\sum_{i=1}^{2^n} (f_{i,left}^{min} + f_{i,right}^{min}) \ge \left(\frac{b-a}{2^n}\right)\sum_{i=1}^{2^n} f_i^{min}$$

Notice that the left side of the above inequality $\left(\frac{b-a}{2^{n+1}}\right)\sum_{i=1}^{2^n}(f_{i,left}^{min} + f_{i,right}^{min}) = \mathcal{L}_{2^{n+1}}$ and the right hand side $\left(\frac{b-a}{2^n}\right)\sum_{i=1}^{2^n} f_i^{min} = \mathcal{L}_{2^n}$, thus we have

$$\mathcal{L}_{2^{n+1}} \ge \mathcal{L}_{2^n},$$

i.e. the sequence $\{\mathcal{L}_{2^n}\}_{n=1}^{\infty}$ is monotone increasing. □

Similarly, we can prove that for the dyadic sequence of partitions of $[a,b]$, the sequence $\{\mathcal{U}_{2^n}\}_{n=1}^{\infty}$ is monotone decreasing.

Theorem 1. *The limits* $\lim_{n\to\infty} \mathcal{L}_{2^n}$ *and* $\lim_{n\to\infty} \mathcal{U}_{2^n}$ *exist.*

Proof. We have proven that the sequences $\{\mathcal{L}_{2^n}\}_{n=1}^{\infty}$ and $\{\mathcal{U}_{2^n}\}_{n=1}^{\infty}$ are bounded and monotone, thus we conclude from the monotone convergence theorem that the sequences converge. □

Now we would like to show that $\lim_{n\to\infty}\{\mathcal{L}_{2^n}\}_{n=1}^{\infty} = \lim_{n\to\infty}\{\mathcal{U}_{2^n}\}_{n=1}^{\infty}$. Why? because we already know that $\mathcal{L}_{2^n} \leq \mathcal{R}_{2^n} \leq \mathcal{U}_{2^n}$, thus if $\lim_{n\to\infty} \mathcal{L}_{2^n} = \lim_{n\to\infty} \mathcal{U}_{2^n} = I$, then we can conclude by the squeeze theorem that $\lim_{n\to\infty} \mathcal{R}_{2^n} = I$.

Lemma (The small-span lemma). *If f is a continuous function on $[a, b]$, then for any $\epsilon > 0$ there exists a real number $\delta > 0$ such that whenever $\|\mathcal{P}\| < \delta$, then $f_i^{max} - f_i^{min} < \epsilon$ on any i^{th} subinterval of $\mathcal{P}$.*

Proof. If f is continuous on the closed and bounded interval $[a, b]$, then by the Heine-Cantor theorem f is uniformly continuous on $[a, b]$. Let $\epsilon > 0$ be given. As f is uniformly continuous, there exists $\delta > 0$ such that

$$\begin{aligned}&\text{whenever } |x_1 - x_2| < \delta\\ &\text{thenever } |f(x_1) - f(x_2)| < \epsilon.\end{aligned}$$

Given $\delta > 0$ we can always find a partition $\mathcal{P}$ of $[a, b]$ with $\|\mathcal{P}\| < \delta$. It follows that the length of any i^{th} subinterval i.e. $|x_i - x_{i-1}| \leq \|\mathcal{P}\| < \delta$. We know that $|x_i^{max} - x_i^{min}| \leq |x_i - x_{i-1}|$, hence $|x_i^{max} - x_i^{min}| < \delta$. Thus, as a consequence of uniform continuity of f, we get $|f(x_i^{max}) - f(x_i^{min})| = f_i^{max} - f_i^{min} < \epsilon$ for any i^{th} subinterval. □

Remark. *The small-span lemma can also be stated in terms of* ***dyadic partitions****: If f is a continuous function on $[a, b]$, then for any $\epsilon > 0$ there exists an integer N and therefore such that whenever $n > N$, thenever $f_i^{max} - f_i^{min} < \epsilon$ on any i^{th} subinterval of $\mathcal{P}_{2^n}$.*

Lemma 3. *For a sequence $\mathcal{P}_{2^n}$ of dyadic partitions of $[a, b]$*

$$\lim_{n\to\infty} (\mathcal{U}_{2^n} - \mathcal{L}_{2^n}) = 0.$$

Proof. Notice, that $\mathcal{U}_{2^n} - \mathcal{L}_{2^n} \leq \left(\frac{b-a}{2^n}\right) \sum_{i=1}^{2^n} f_i^{max} - f_i^{min}$.
To prove that $\lim_{n\to\infty} \mathcal{U}_{2^n} - \mathcal{L}_{2^n} = 0$, we need to show that for any $\epsilon > 0$ there exists an integer N such that $\mathcal{U}_{2^n} - \mathcal{L}_{2^n} < \epsilon$ for all $n > N$.
By the small-span lemma, for any ϵ^* there exists an integer N such that for all i and for all $n > N$.

$$\begin{aligned}
f_i^{max} - f_i^{min} &< \epsilon^* \\
\sum_{i=1}^{2^n} f_i^{max} - f_i^{min} &< \sum_{i=1}^{2^n} \epsilon^* \\
\left(\frac{b-a}{2^n}\right) \sum_{i=1}^{2^n} f_i^{max} - f_i^{min} &< \left(\frac{b-a}{2^n}\right) \sum_{i=1}^{2^n} \epsilon^* = \left(\frac{b-a}{2^n}\right) 2^n\, \epsilon^* = (b-a)\epsilon^* \\
\mathcal{U}_{2^n} - \mathcal{L}_{2^n} &< (b-a)\epsilon^*.
\end{aligned}$$

This is true for any $\epsilon^* > 0$, hence it is true for $\epsilon^* = \frac{\epsilon}{b-a}$. Thus, for any $\epsilon > 0$ there exists an integer N such that $|\mathcal{U}_{2^n} - \mathcal{L}_{2^n}| < \epsilon$ for all $n > N$, i.e. $\lim_{n\to\infty}(\mathcal{U}_{2^n} - \mathcal{L}_{2^n}) = 0$. □

Remark 1. *As $\lim_{n\to\infty} \mathcal{L}_{2^n}$ and $\lim_{n\to\infty} \mathcal{U}_{2^n}$ both exist, and $\lim_{n\to\infty}(\mathcal{U}_{2^n} - \mathcal{L}_{2^n}) = 0$, we can conclude that $\lim_{n\to\infty} \mathcal{L}_{2^n} = \lim_{n\to\infty} \mathcal{U}_{2^n}$.*
Let us denote this limit by I, i.e. $\lim_{n\to\infty} \mathcal{L}_{2^n} = I$.

Moreover, $\mathcal{L}_{2^n} \leq \mathcal{R}_{2^n} \leq \mathcal{U}_{2^n}$, and $\lim_{n\to\infty} \mathcal{L}_{2^n} = \lim_{n\to\infty} \mathcal{U}_{2^n} = I$;
we can conclude $\lim_{n\to\infty} \mathcal{R}_{2^n} = I$.

5. Results related to the general partition $\mathcal{P}$

Let $\mathcal{P}$ be a general partition on $[a, b]$.

Lemma 4.

$$\lim_{\|\mathcal{P}\|\to 0} (\mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}}) = 0.$$

Proof. To show that $\lim_{\|\mathcal{P}\|\to 0}(\mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}}) = 0$, we need show that for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $|\|\mathcal{P}\| - 0| < \delta$ i.e. whenever $\|\mathcal{P}\| < \delta$, thenever $\mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}} < \epsilon$. Let us look at $\mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}}$.

$$\mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}} = \sum_{i=1}^{n} (f_i^{max} - f_i^{min}) \triangle x_i$$

Let $\epsilon > 0$ be given. Due to the uniform continuity of f we can find a number $\delta > 0$ for any $\epsilon^* > 0$, such that whenever $\|\mathcal{P}\| < \delta$, thenever $f_i^{max} - f_i^{min} < \epsilon^*$, for any i^{th} subinterval in $\mathcal{P}$. Notice also that $\sum_{i=1}^{n} \triangle x_i = b - a$. Thus,

$$\mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}} = \sum_{i=1}^{n} (f_i^{max} - f_i^{min}) \triangle x_i < \sum_{i=1}^{n} \epsilon^* \triangle x_i = \epsilon^* \sum_{i=1}^{n} \triangle x_i = \epsilon^*(b - a)$$

$$\text{Thus,} \quad \mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}} < \epsilon^*(b - a).$$

This is true for any $\epsilon^* > 0$, thus we set $\epsilon^* = \frac{\epsilon}{b-a}$. Hence, given $\epsilon > 0$ we have found a number $\delta > 0$ such that whenever $\|\mathcal{P}\| < \delta$, thenever $\mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}} < \epsilon$, i.e. $\lim_{\|\mathcal{P}\|\to 0}(\mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}}) = 0$. □

Lemma 5. *Let $\mathcal{P}'$ be a refinement of $\mathcal{P}$ i.e. $\mathcal{P}'$ is obtained by adding more points to $\mathcal{P}$. Then $\mathcal{L}_{\mathcal{P}} \leq \mathcal{L}_{\mathcal{P}'} \leq \mathcal{U}_{\mathcal{P}'} \leq \mathcal{U}_{\mathcal{P}}$.*

Proof. $\mathcal{L}_{\mathcal{P}'} \leq \mathcal{U}_{\mathcal{P}'}$ is true by definition. The proof of $\mathcal{L}_{\mathcal{P}} \leq \mathcal{L}_{\mathcal{P}'}$ is very similar to the proof of $\mathcal{L}_{2^n} \leq \mathcal{L}_{2^{n+1}}$. This is left to the reader. Similarly, $\mathcal{U}_{\mathcal{P}'} \leq \mathcal{U}_{\mathcal{P}}$ also follows. □

Remark. *$\mathcal{L}_{\mathcal{P}'} \leq \mathcal{R}_{\mathcal{P}'} \leq \mathcal{U}_{\mathcal{P}'}$ by definition, and $\mathcal{L}_{\mathcal{P}} \leq \mathcal{R}_{\mathcal{P}} \leq \mathcal{U}_{\mathcal{P}}$, also by definition. Thus, it is true that*

$$\begin{array}{lllll} \mathcal{L}_{\mathcal{P}} \leq \mathcal{L}_{\mathcal{P}'} & \leq \mathcal{R}_{\mathcal{P}'} & \leq \mathcal{U}_{\mathcal{P}'} & \leq \mathcal{U}_{\mathcal{P}} \\ \mathcal{L}_{\mathcal{P}} & \leq \mathcal{R}_{\mathcal{P}} \leq & & \mathcal{U}_{\mathcal{P}} \end{array}$$

Theorem 2.

$$\lim_{\|\mathcal{P}\|\to 0} |\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}'}| = 0.$$

Proof. We see from the previous remark, that both $\mathcal{R}_{\mathcal{P}}$ and $\mathcal{R}_{\mathcal{P}'}$ lie somewhere between $\mathcal{L}_{\mathcal{P}}$ and $\mathcal{U}_{\mathcal{P}}$. Thus, $|\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}'}| < \mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}}$. But from the lemma **??** we have $\lim_{\|\mathcal{P}\|\to 0}(\mathcal{U}_{\mathcal{P}} - \mathcal{L}_{\mathcal{P}}) = 0$, thus we conclude that $\lim_{\|\mathcal{P}\|\to 0} |\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}'}| = 0$. □

6. The final stroke

Theorem 3.

$$\lim_{\|\mathcal{P}\| \to 0} \mathcal{R}_{\mathcal{P}} = I.$$

Proof. We want to show that given any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\|\mathcal{P}\| < \delta$ thenever $|\mathcal{R}_{\mathcal{P}} - I| < \epsilon$.
Let us first fix $\epsilon > 0$, and now look at $|\mathcal{R}_{\mathcal{P}} - I|$.

$$\begin{aligned}
|\mathcal{R}_{\mathcal{P}} - I| &= |\mathcal{R}_{\mathcal{P}} - \mathcal{L}_{2^n} + \mathcal{L}_{2^n} - I| \\
&\leq |\mathcal{R}_{\mathcal{P}} - \mathcal{L}_{2^n}| + |\mathcal{L}_{2^n} - I| \qquad \text{(triangle inequality)} \\
&= |\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}'} + \mathcal{R}_{\mathcal{P}'} - \mathcal{L}_{2^n}| + |\mathcal{L}_{2^n} - I| \\
&\leq |\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}'}| + |\mathcal{R}_{\mathcal{P}'} - \mathcal{L}_{2^n}| + |\mathcal{L}_{2^n} - I| \qquad \text{(triangle inequality)}
\end{aligned}$$

(1) If $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then given any $\epsilon_1 > 0$, we can find $\delta_1 > 0$ such that whenever $\|\mathcal{P}\| < \delta_1$ thenever $|\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}'}| < \epsilon_1$. Let us choose $\epsilon_1 = \frac{\epsilon}{3}$. Thus, there exists $\delta_1 > 0$ such that whenever $\|\mathcal{P}\| < \delta_1$ thenever $|\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}'}| < \frac{\epsilon}{3}$.

(2) Note that $\mathcal{P}_{2^n}$ is a special type of partition $\mathcal{P}$, and thus $\mathcal{L}_{2^n}$ is a special type of Riemann sum $\mathcal{R}_{\mathcal{P}}$. Hence, if $\mathcal{P}'$ is **also** a refinement of $\mathcal{P}_{2^n}$. Thus, given $\epsilon_2 = \frac{\epsilon}{3}$ there exists $\delta_2 > 0$ such that whenever $\|\mathcal{P}_{2^n}\| = \frac{b-a}{2^n} < \delta_2$, thenever $|\mathcal{R}_{\mathcal{P}'} - \mathcal{L}_{2^n}| < \frac{\epsilon}{3}$.

    Note, that we need both (1) and (2), therefore we need $\mathcal{P}'$ to be a refinement of both $\mathcal{P}$ and $\mathcal{P}_{2^n}$, i.e. $\mathcal{P}'$ contains all points of $\mathcal{P}$ and all point of $\mathcal{P}_{2^n}$ and few more.

(3) As $I = \lim_{n\to\infty} \mathcal{L}_{2^n}$, by definition of limits, for $\epsilon_3 = \frac{\epsilon}{3}$ there exist N_3 such that $|\mathcal{L}_{2^n} - I| < \frac{\epsilon}{3}$ for all $n > N_3$ i.e. for all $\|\mathcal{P}_{2^n}\| = \frac{b-a}{2^n} < \delta_3 = \frac{1}{2^{N_3}}$.

We want (1), (2) and (3) to be true simultaneously. Thus we replace δ_1, δ_2 and δ_3 with the minimum of three values, $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Thus all 3 inequalities will be true for this δ. So, whenever $\|\mathcal{P}\| < \delta$, thenever $|\mathcal{R}_{\mathcal{P}} - I| \leq |\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}'}| + |\mathcal{R}_{\mathcal{P}'} - \mathcal{L}_{2^n}| + |\mathcal{L}_{2^n} - I| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. ∎

DONE!