ENERGY METHODS IN IMAGE PROCESSING WITH EDGE ENHANCEMENT

PRASHANT ATHAVALE†

Abstract. Digital images are can be realized as $L^2(R^2)$ objects. Noise is introduced in a digital image due to various reasons. Some of the reasons are the limitations of the image capturing device, e.g. improper lens adjustments, blurring due to relative motion between the camera and the object. Noise can also be added during the signal transmission. Thus the observed image f deviates from the original image u and we write $f = Ru + \eta$, where R represents a linear blurring operator and η denotes an additive noise.

Various variational methods are used in order to recover the original image u. In particular we will look at the problem of minimizing following energy functional

$$E(u) = \int_\Omega \phi(|\nabla u|) + \lambda \int_\Omega |f - Ru|^2.$$

When the function ϕ is identity, in absence of blurring, this problem is the Rudin-Osher-Fatemi minimization problem. We will discuss the existence and uniqueness of the solution of this minimization problem. The solution is obtained either iteratively or using the gradient descent scheme. If one uses uniform grid for numerical implementation it becomes difficult to simulate sharp edges. In this paper a new approach is presented where the grid depends on the value of the gradient. This can also be written in a form of a partial differential equation where the parameter λ is replaced by a function $\mu = \lambda/g(|G \star \nabla u|)$, where g is a real valued function on $\mathbb{R}^+$ with $g(0) = 1$ and $\lim_{s\to\infty} g(s) = 0$. Numerical experiments are presented in this paper to demonstrate the difference between the traditional uniform grid method and the proposed method.

Key words. natural images, multiscale expansion, total variation, adaptivity, functions of bounded variation

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1. Introduction. A greyscale (black and white) digital image is obtained by sampling and quantization of an analogue image. The image can be degraded due to various reasons like improper focusing of the camera lens, mechanical defects, atmospheric turbulence etc. There could also be some *noise* introduced by errors in signal transmission. A common model is to let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be the clean image. The operator R represents the blurring operator and η represents the additive noise. Hence, the observed image f can be expressed as $f = Ru + \eta$. The problem is to recover the image u. The first idea to recover the image u by minimization of energy was proposed by Tikhonov and Arsenin [13]. They proposed the minimization of the following functional to recover u

$$F(u) = \int_\Omega |\nabla u|^2 + \lambda \int_\Omega |f - Ru|^2 dx. \tag{1.1}$$

The first the smoothing term which forces the gradient to be small, thus reducing the noise. The second term is the fidelity term. The Euler-Lagrange differential equation for minimizing (1.1) is as follows:

$$\nabla u + \lambda(R^* f - R^* Ru) = 0$$

†Department of Mathematics, Center of Scientific Computation and Mathematical Modeling (CSCAMM), University of Maryland, College Park, MD 20742 (prashant@cscamm.umd.edu).

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

But the Laplacian operator is an isotropic smoothing operator which smooths the edges along with the noise. This is undesirable.

Rudin, Osher, and Fatemi [10] considered minimizing the following functional

$$J(u) = \int_\Omega |\nabla u| + \lambda \int_\Omega (f-u)^2 dx. \tag{1.2}$$

The $v = f - u$ part is given a lot of attention, from the modern view point. This part contains not only the noise but also the edges and the texture, which are important for any image.

Aubert and Vese [4] studied minimization of the following functional, which was originally proposed by Geman and Geman [7].

$$E(u) = \int_\Omega \phi(|\nabla u|) + \lambda \int_\Omega |f - Ru|^2. \tag{1.3}$$

This approach was earlier taken by Geman and Geman [7].

2. Minimizing the functional $E(u)$. The natural space on which we would be able to seek a solution is the space $V = \{u \in L^2(\Omega), \nabla u \in L^1(\Omega)^2\}$. Unfortunately this space is not reflexive [3]. Hence we have to consider the relaxed functional $\tilde{E}$ instead of E, given by

$$\tilde{E}(u) = \int_\Omega \phi(|\nabla u|) + c\int_{S_u} (u^+ - u_-) d\mathcal{H}^1 + c\int_{\Omega - S_u} |C_u| + \int_\Omega |u_0 - Ru|^2,$$

where $c = \lim_{s\to+\infty} \frac{\phi(s)}{s}$.

Existence and uniqueness of the minimizer of $\tilde{E}(u)$: Let us discuss the existence and uniqueness of the minimizer of the $\tilde{E}(u)$ in the space BV i.e. the space of bounded variations [2], [14]. We shall follow the treatment in [3]. The existence of the minimizer of $\tilde{E}(u)$ is proved by first showing that a minimizing sequence u_n is bounded in $BV(\Omega)$ and thus there exists a u_0 in $BV(\Omega)$ such that $u_n \underset{BV-w^*}{\longrightarrow} u_0$ and $Ru_n \underset{L^2(\Omega)}{\longrightarrow} Ru_0$. Finally from the weak semicontinuity property of the convex function of measures and the weak semicontinuity of the L^2 norm, we get

$$\int_\Omega |Ru_0 - f|^2 \leqslant \liminf_{n\to+\infty} \int_\Omega |Ru_n - f|^2$$
$$\int_\Omega \phi(|\nabla u_0|) \leqslant \liminf_{n\to+\infty} \int_\Omega \phi(|\nabla u_n|).$$

That is,

$$\tilde{E}(u_0) \leqslant \liminf_{n\to+\infty} = \inf_{u\in BV(\Omega)} \tilde{E}(u).$$

i.e. u_0 is a minimum point of $\tilde{E}(u)$.
To prove the uniqueness of the minimizer we assume that the $R.1 \neq 0$. Let u_0 and v_0 be two minima of $\tilde{E}(u)$. Because of the convexity of $E(u)$ we get that $\nabla u_0 = \nabla v_0$ and $Ru_0 = Rv_0$. Hence, $u = v + c$. This implies that $Rc = 0$. This is contradiction to our assumption that $R.1 \neq 0$. Hence, $c = 0$ i.e. $u_0 = v_0$.

3. Selection of the function ϕ. The Euler Lagrange equation for minimization of $E(u)$ in (1.3) is as follows

$$-\operatorname{div}\left(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u\right) + 2\lambda(R^*Ru - R^*f) = 0 \tag{3.1}$$

Now for each point $x \in \Omega$ where $|\nabla u(x)| \neq 0$ one can define the vectors $N(x)$ and $T(x)$ as

$$N(x) = \frac{\nabla u(x)}{|\nabla u(x)|} \text{ and } T(x) = \frac{(u_y(x), -u_x(x))}{|\nabla u(x)|}.$$

The vectors $N(x)$ and $T(x)$ are respectively the normal and tangent to the level curves at x. Let $\nabla^2 u$ denote the Hessian matrix i.e.

$$\nabla^2 u = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}.$$

Let u_{NN} and u_{TT} denote the second derivatives of u in the direction of $N(x)$ and $T(x)$, i.e.

$$u_{NN} = \langle N, \nabla^2 u N\rangle = \frac{1}{|\nabla u|^2}(u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy})$$
$$u_{TT} = \langle T, \nabla^2 u T\rangle = \frac{1}{|\nabla u|^2}(u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx}).$$

Using this notation one can write the Euler Lagrange differential equation as

$$-\left(\frac{\phi'(|\nabla u|)}{|\nabla u|}u_{TT} + \phi''(|\nabla u|)u_{NN}\right) + 2\lambda(R^*Ru - R^*f) = 0. \tag{3.2}$$

Writing the divergence term $\operatorname{div}\left(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u\right)$ as a weighted sum of two directional derivatives along N and T allows us to see clearly the action of the diffusion operator in the direction of T and N. This also helps in deciding the properties that the function ϕ must have. To this effect we follow [3].

3.1. Conditions on ϕ at low gradients. When the image has a low gradients i.e. it at a point $x \in \Omega$ the gradient $|\nabla u|$ is small , then we would like to have a uniform smoothing in all directions. Thus we need the coefficients of u_{TT} and u_{NN} to be equal as $|\nabla u| \to 0^+$. Hence, we have the following condition

$$\lim_{s\to 0^+} \frac{\phi'(s)}{s} = \lim_{s\to 0^+} \phi''(s) = \phi''(0) > 0.$$

Thus if $|\nabla u| \to 0^+$ the (3.2) becomes

$$-\phi''(0)(u_{TT} + u_{NN}) + 2\lambda(R^*Ru - R^*f) = 0.$$

But note that $u_{TT} + u_{NN} = u_{xx} + u_{yy} = \Delta u$. Hence we get,

$$-\phi''(0)\Delta u + 2\lambda(R^*Ru - R^*f) = 0.$$

This is a uniformly elliptic equation with strong isotropic smoothing properties. We also impose that the function ϕ is relatively insensitive to small changes in the gradient of the image. Thus we also impose $\lim_{s\to 0^+} \phi'(s) = 0$.

3.2. Conditions on ϕ near edges. If there is an edge at $x \in \Omega$ we do not want isotropic smoothing. In fact we desire smoothing only in the direction of $T(x)$ and no smoothing in the direction of $N(x)$. Thus in (3.2) we need the coefficient of u_{NN} to be zero and the coefficient of u_{TT} to be some positive number β. i.e. we have the following conditions on ϕ.

$$\lim_{s\to\infty} \phi''(s) = 0 \tag{3.3}$$

$$\lim_{s\to\infty} \frac{\phi'(s)}{s} = \beta > 0. \tag{3.4}$$

Unfortunately, the conditions (3.3) and (3.4) are not compatible with each other. Hence usually a compromise is found.

4. Rudin Osher Fatemi model (ROF model). The first model to consider the minimization of the total variation was introduced by Rudin, Osher and Fatemi. They considered minimizing the functional $J(u)$ which is re-written here for convenience.

$$J(u) = \int_\Omega |\nabla u| + \lambda \int_\Omega (f-u)^2 dx.$$

The functional $E(u)$ in (1.3) is a generalized version of the ROF model. So the analysis of functional $E(u)$ applies to the $J(u)$ in (1.2) also. The Euler Lagrange differential equation for minimization of $J(u)$ is as follows

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + 2\lambda(u-f) = 0. \tag{4.1}$$

If we denote $v = f - u$, this minimization can be considered as a decomposition of f into u and v. Multiscale image decomposition was proposed by Tadmor, Nezzar and Vese [11], [12], where they achieve multiscale image decomposition by decomposing the v part again by doubling the λ. Where the u part contains homogeneous regions and the v part contents textures and noise. The ROF decomposition was studied by Y. Meyer in [9]. He introduced the Banach space G and the associated norm as follows.

Definition 4.1. *$G(\Omega)$ is the subspace of $W^{-1,\infty}(\Omega)$ defined by*

$$G(\Omega) = \{v \in L^2(\Omega) : v = \operatorname{div}(g), g \in L^\infty(\Omega),\ g\cdot N = 0 \text{ on } \partial\Omega\}.$$

The subspace $G(\Omega)$ can be endowed with the norm

$$\|v\|_* = \inf\{\|g\|_{L^\infty(\Omega)} : v = \operatorname{div}(g),\ g\cdot N = 0 \text{ on } \partial\Omega\}.$$

Meyer [9] also showed an interesting connection between the the G-space and the space of bounded variations i.e. the BV space. He proved the following theorem which also appears in [5].

THEOREM 4.2. *If f, u, v are three functions in $L^2(\mathbb{R}^2)$ and if $\|f\| > \frac{1}{2\lambda}$, then the Rudin-Osher-Fatemi decomposition $f = u + v$ is characterized by the following two conditions*

$$\|v\|_* = \frac{1}{2\lambda} \text{ and } \int u(x)v(x) = \frac{1}{2\lambda}\|u\|_{BV}.$$

We follow [9] to define the extremal pair u and v as follows.

DEFINITION 4.3. *Let u and v be real valued functions in $L^2(\mathbb{R}^2)$ and $u \in BV$. We say that (u, v) is an extremal pair if*

$$\int u(x)v(x)dx = \|u\|_{BV}\|v\|_*.$$

The term $\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)(x_0)$ is a curvature at a point x_0 of a level curve of u at x_0, up to a sign factor. It is easy to observe the following corollary.

COROLLARY 4.4. *With Neumann boundary conditions on u the star-norm of the function $w = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ is unity.*

Proof. Let $w = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ and φ be a test function. Thus,

$$\begin{aligned} |\int_\Omega w\varphi| &= |\int_\Omega \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\varphi| \\ &= |-\int_\Omega \frac{\nabla u}{|\nabla u|}\cdot\nabla\varphi| \\ &\leqslant \int_\Omega |\nabla\varphi| \\ \frac{|\int_\Omega w\varphi|}{\int_\Omega|\nabla\varphi|} &\leqslant 1 \\ \text{Thus, } \|w\|_* &\leqslant 1. \end{aligned} \tag{4.2}$$

Also taking $\varphi = u$ one gets,

$$\begin{aligned} |\int_\Omega uv| &= \int_\Omega |\nabla u| \\ \frac{|\int_\Omega uv|}{\int_\Omega |\nabla u|} &= 1 \\ \text{Thus, } \|w\|_* &\geqslant \frac{|\int_\Omega uv|}{\int_\Omega |\nabla u|} = 1. \end{aligned} \tag{4.3}$$

Using (4.2) and (4.3) we see that $\|w\|_* = 1$. □

5. Solving the ROF model. The Euler Lagrange differential equation (4.1) corresponding to the ROF model (1.2) can be written as

$$u = f + \frac{1}{2\lambda}\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \text{ in } \Omega. \tag{5.1}$$

On the boundary we impose the Neumann boundary condition, i.e. $\frac{\partial u}{\partial \nu} = 0$ on the boundary $\partial\Omega$. Numerical implementation can be done using fixed point iteration as in [11]. The functional (1.2) is replaced by its regularized form

$$J^{\varepsilon}(u) = \int_{\Omega} \sqrt{\varepsilon^2 + |\nabla u|^2} + \lambda \int_{\Omega} (f-u)^2 dx.$$

The associated Euler Lagrange differential equations for this regularized functional reads

$$u = f + \frac{1}{2\lambda} \operatorname{div}\Big(\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}}\Big) \text{ in } \Omega \tag{5.2}$$
$$\text{with } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

The region Ω is covered with computational grid $(x_i, y_j) = (ih, jh)$ where h is a cell size. Let $D_+ = D_+(h)$, $D_- = D_-(h)$, and $D_0 := (D_+ + D_-)/2$ denote the usual forward, backward, and centered divided difference.

Thus, $D_{+x}u_{i,j} = (u_{i+1,j} - u_{i,j})/h$, $D_{-x}u_{i,j} = (u_{i,j} - u_{i-1,j})/h$, $D_{+y}u_{i,j} = (u_{i,j+1} - u_{i,j})/h$, $D_{-y}u_{i,j} = (u_{i,j} - u_{i,j-1})/h$, $D_{0x}u_{i,j} = (u_{i+1,j} - u_{i-1,j})/2h$ and $D_{0y}u_{i,j} = (u_{i,j+1} - u_{i,j-1})/2h$. With this notation (5.2) can be discretized as follows.

$$\begin{aligned} u_{i,j} &= f_{i,j} + \frac{1}{2\lambda} D_{-x}\Big[\frac{D_{+x}u_{i,j}}{\sqrt{\varepsilon^2 + (D_{+x}u_{i,j})^2 + (D_{0y}u_{i,j})^2}}\Big] \\ &\quad + \frac{1}{2\lambda} D_{-y}\Big[\frac{D_{+y}u_{i,j}}{\sqrt{\varepsilon^2 + (D_{0x}u_{i,j})^2 + (D_{+y}u_{i,j})^2}}\Big] \\ &= f_{i,j} + \frac{1}{2\lambda h^2}\Big[\frac{u_{i+1,j} - u_{i,j}}{\sqrt{\varepsilon^2 + (D_{+x}u_{i,j})^2 + (D_{0y}u_{i,j})^2}} - \frac{u_{i,j} - u_{i-1,j}}{\sqrt{\varepsilon^2 + (D_{-x}u_{i,j})^2 + (D_{0y}u_{i-1,j})^2}}\Big] \\ &\quad + \frac{1}{2\lambda h^2}\Big[\frac{u_{i,j+1} - u_{i,j}}{\sqrt{\varepsilon^2 + (D_{0x}u_{i,j})^2 + (D_{+y}u_{i,j})^2}} - \frac{u_{i,j} - u_{i,j-1}}{\sqrt{\varepsilon^2 + (D_{0x}u_{i,j-1})^2 + (D_{-y}u_{i,j})^2}}\Big]. \end{aligned}$$

The fixed point iteration can be written as follows

$$\begin{aligned} u_{i,j}^{n+1} &= f_{i,j} + \frac{1}{2\lambda h^2}\Big[\frac{u_{i+1,j}^{n} - u_{i,j}^{n+1}}{\sqrt{\varepsilon^2 + (D_{+x}u_{i,j}^{n})^2 + (D_{0y}u_{i,j}^{n})^2}} - \frac{u_{i,j}^{n+1} - u_{i-1,j}^{n}}{\sqrt{\varepsilon^2 + (D_{-x}u_{i,j}^{n})^2 + (D_{0y}u_{i-1,j}^{n})^2}}\Big] \\ &\quad + \frac{1}{2\lambda h^2}\Big[\frac{u_{i,j+1}^{n} - u_{i,j}^{n+1}}{\sqrt{\varepsilon^2 + (D_{0x}u_{i,j}^{n})^2 + (D_{+y}u_{i,j}^{n})^2}} - \frac{u_{i,j}^{n+1} - u_{i,j-1}^{n}}{\sqrt{\varepsilon^2 + (D_{0x}u_{i,j-1}^{n})^2 + (D_{-y}u_{i,j}^{n})^2}}\Big]. \end{aligned}$$

Now let us use the notation $\Delta_{+x}u_{i,j} = (u_{i+1,j} - u_{i,j})$, $\Delta_{-x}u_{i,j} = (u_{i,j} - u_{i-1,j})$, $\Delta_{+y}u_{i,j} = (u_{i,j+1} - u_{i,j})$, $\Delta_{-y}u_{i,j} = (u_{i,j} - u_{i,j-1})$, $\Delta_{0x}u_{i,j} = (u_{i+1,j} - u_{i-1,j})/2$ and $\Delta_{0y}u_{i,j} = (u_{i,j+1} - u_{i,j-1})/2$ in the above equation to get the following.

$$\begin{aligned} u_{i,j}^{n+1} &= f_{i,j} + \frac{1}{2\lambda h}\Big[\frac{u_{i+1,j}^{n} - u_{i,j}^{n+1}}{\sqrt{(\varepsilon h)^2 + (\Delta_{+x}u_{i,j}^{n})^2 + (\Delta_{0y}u_{i,j}^{n})^2}} - \frac{u_{i,j}^{n+1} - u_{i-1,j}^{n}}{\sqrt{(\varepsilon h)^2 + (\Delta_{-x}u_{i,j}^{n})^2 + (\Delta_{0y}u_{i-1,j}^{n})^2}}\Big] \\ &\quad + \frac{1}{2\lambda h}\Big[\frac{u_{i,j+1}^{n} - u_{i,j}^{n+1}}{\sqrt{(\varepsilon h)^2 + (\Delta_{0x}u_{i,j}^{n})^2 + (\Delta_{+y}u_{i,j}^{n})^2}} - \frac{u_{i,j}^{n+1} - u_{i,j-1}^{n}}{\sqrt{(\varepsilon h)^2 + (\Delta_{0x}u_{i,j-1}^{n})^2 + (\Delta_{-y}u_{i,j}^{n})^2}}\Big]. \end{aligned}$$

Now let us use the following notation.

$$c_E \equiv \frac{1}{\sqrt{(\varepsilon h)^2 + (\Delta_{+x} u^n_{i,j})^2 + (\Delta_{0y} u^n_{i,j})^2}}, c_W \equiv \frac{1}{\sqrt{(\varepsilon h)^2 + (\Delta_{-x} u^n_{i,j})^2 + (\Delta_{0y} u^n_{i-1,j})^2}},$$

$$c_S \equiv \frac{1}{\sqrt{(\varepsilon h)^2 + (\Delta_{0x} u^n_{i,j})^2 + (\Delta_{+y} u^n_{i,j})^2}}, c_N \equiv \frac{1}{\sqrt{(\varepsilon h)^2 + (\Delta_{0x} u^n_{i,j-1})^2 + (\Delta_{-y} u^n_{i,j})^2}}.$$

With this notation the fixed point iteration can be written as

$$u^{n+1}_{i,j} = \frac{2\lambda h f_{i,j} + c_E u^n_{i+1,j} + c_W u^n_{i-1,j} + c_S u^n_{i,j+1} + c_N u^n_{i,j-1}}{2\lambda h + c_E + c_W + c_S + c_N}. \tag{5.3}$$

6. Role of 'h' in numerical computation. In image processing there is always the question of what the distance between the pixels should be taken for an image of pixel-size $(M \times M)$. Some authors use $h = 1$, and the others use $h = 1/M$. Taking a different value of h results in different results in image processing algorithms whenever numerical derivatives are used. The reason this happens is due to the fact that the numerical derivatives change with the change in h.

If the image is of pixel-size (256×256) and the image is a grey scale 8-bit image, i.e. it has $2^8 = 256$ intensity levels [8], then it makes sense to take $h = 1$. Since, in that case each row (and each column) of the image f would be a mapping $f_{row} : K \to K$, where K is a closed interval $[0, 256]$.

So, one way is to take $h = 1$ always. But now the problem comes when we try these algorithms on a scaled version of an image i.e. image of lenna which is (256×256) pixel-size image of lenna and (512×512) pixel-size image of lenna. It is desirable for any algorithm to have the same effect on the image irrespective of its size i.e. the results from differently scaled images should be scaled versions of each other. This will happen if the the domain of the image f is always $\Omega = [0, 2^b] \times [0, 2^b]$. Hence, for an image of pixel-size $(M \times M)$ we should take $h = \frac{2^b}{M}$. But one should note that the selection of different cell size, which is uniform throughout the image amounts to only scaling effect.

7. New method proposed for images with sharp edges. At points of discontinuities the derivative of the function u is undefined. But the numerical approximation of the (5.2) assigns a finite values to derivatives at sharp edges. For example let the function y be defined as follows.

$$y(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

The derivative of $y(x)$ at $x = 0$ is the dirac delta function $\delta(x)$ which is a measure. Besides, we need to find div $\left(\frac{\nabla u}{|\nabla u|}\right)$. This suggests that we should not let the numerical gradient $\nabla_h u$ grow very large. This can be achieved by taking large value of the cell size when we see large value of the numerical gradient $\nabla_h u$. i.e. we should modify the cell size h to $\hbar$ depending on the gradient $\nabla_h u$ for some fixed h. Thus we let the new cell size $\hbar$ to be equal to $h/g(|G \star \nabla_h u|)$ for some real valued function g on $\mathbb{R}^+$. As discussed in the previous section we may select this fixed h as $\frac{2^b}{M}$. We denote by $|G \star \nabla_h u|$ the smoothed version of the numerical gradient $\nabla_h u$ obtained by convolving it with some smoothing kernel G. The function g should be chosen such that $\hbar \approx h$ when the gradient $\nabla_h u$ is low and $\hbar \to \infty$ if $\nabla_h u \to \infty$. Thus, we need the function g to have the following properties

$$g(0) = 1 \text{ and}$$
$$\lim_{s\to\infty} g(s) = 0.$$

There are many functions which satisfy these properties. We can select $g(s) = 1/(1+s^p)$ for some $p > 0$. For 8-bit greyscale image of size 256×256 we get $\hbar = 1+|G\star\nabla_h u|^p$.

With this notation the fixed point iteration (5.3) is an approximation of the following partial differential equation

$$u = f + \frac{g(|G\star\nabla u|)}{2\lambda}\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \text{ in } \Omega.$$

Which can be written as follows

$$u = f + \frac{1}{2\mu}\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \text{ in } \Omega. \tag{7.1}$$

Where, $\mu = \frac{\lambda}{g(|G\star\nabla u|)}$. The function $g(|G\star\nabla u|)$ can be seen as a function that controls the diffusion just as in the work by Alvarez et. al. [1]. This approach can be generalized for (3.1) which can be modified as

$$R^*Ru = R^*f + \frac{g(|G\star\nabla u|)}{2\lambda}\operatorname{div}\left(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u\right).$$

On the boundary we impose the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$.

8. Numerical experiments. Numerical experiments were done on 256×256 sized grey scale 8 bit images of a black and white circle and the image of Lenna as shown in Figure 8.1. All the experiments were done with $h = 1$ and $\lambda = 0.0001$. The Figure 8.2 shows the implementation of standard ROF method with uniform grid i.e. with constant cell size $h = 1$. The image on the left is the u part of the original image. The image on the right is the v part. The Figure 8.3 shows the u part and v part of the image of circle, obtained with the new proposed method i.e. using (7.1). Note that the v part is identically equal to zero. Similarly, Figure 8.4 shows the implementation of the standard ROF method with uniform grid on the image of Lenna. Compare that to the results obtained with the new proposed method in Figure 8.5. In the numerical experiments, the smoothed gradient $|G\star\nabla u|$ was obtained using the smoothing Gaussian kernel $G(x,y) = \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$ [6]. Here, $\sigma = h$ was used. The function g was taken as $g(s) = \frac{1}{1+s^2}$.

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FIG. 8.1. *Original images used for the experiments:* 256×256 *sized grey scale 8-bit images of Circle and Lenna.*

FIG. 8.2. *The u part and v part of the image of circle in Figure 8.1 using uniform grid with* $h = 1$ *and* $\lambda = 0.0001$.

FIG. 8.3. *The u part and v part of the image of circle in Figure 8.1 using the new proposed partial differential equation (7.1) with* $h = 1$ *and* $\lambda = 0.0001$. *Note that the edge of the circle is well preserved in this experiment and the v part is identically zero.*

Fig. 8.4. *The u part and v part of the image of Lenna in Figure 8.1 using uniform grid with* $h = 1$ *and* $\lambda = 0.0001$*.*

Fig. 8.5. *The u part and v part of the image of Lenna in Figure 8.1 using the new proposed partial differential equation (7.1) with* $h = 1$ *and* $\lambda = 0.0001$*. Notice that the u part in this experiment contains more edges than the standard ROF model.*

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